3.5: Limits at Infinity

There are two types of limits involving infinity.

Limits at infinity, written in the form $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$, are related to horizontal asymptotes.

Infinite limits (covered in Section 1.5) take the form of statements like $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. Infinite limits generally result in vertical asymptotes.

When combined, these two types of limits involving infinity result in statements such as $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = -\infty$, which describe the end behavior of graphs.

Limits at infinity:

Let *f* be a function defined on some interval (a,∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to *L* by taking *x* sufficiently large.

More precisely, $\lim_{x\to\infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number $M > 0$ such that for all *x*, $|f(x) - L| < \varepsilon$ whenever $x > M$.

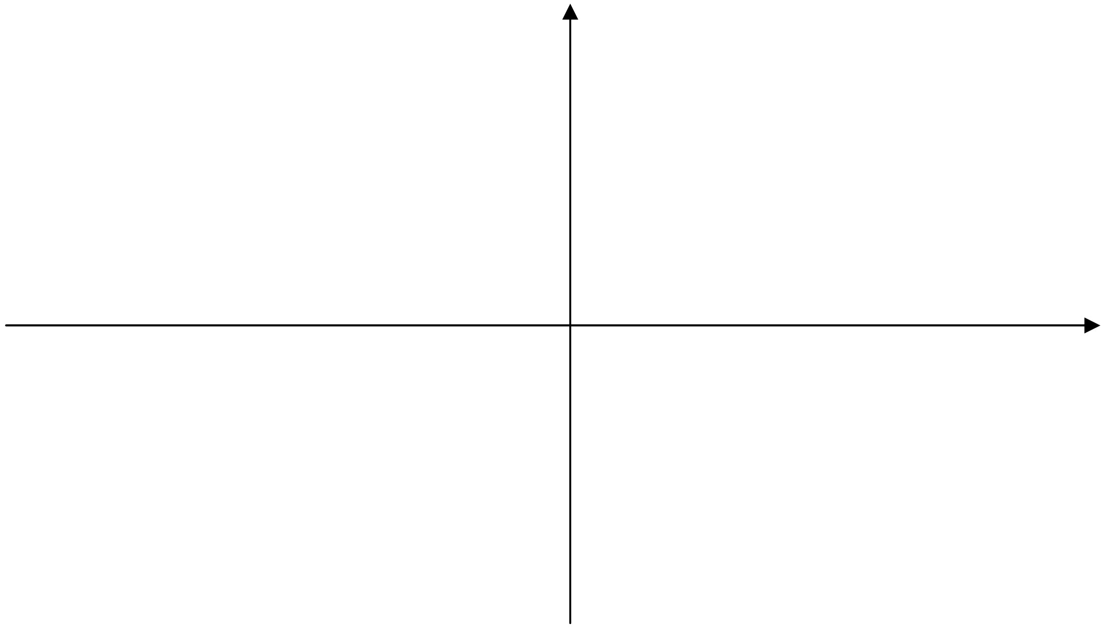
Let *f* be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to *L* by making *x* a sufficiently large negative number.

More precisely, $\lim_{x\to\infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number $N < 0$ such that for all *x*, $|f(x) - L| < \varepsilon$ whenever $x < N$.

Example 1:

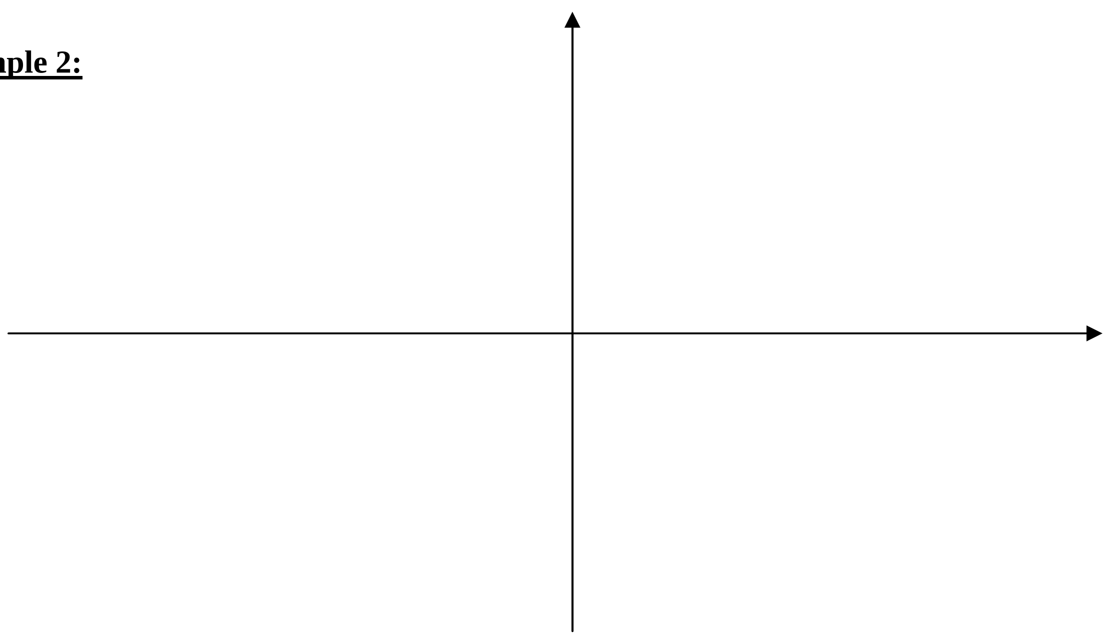


Horizontal asymptotes:

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L \, .$$

Example 2:



Example 3: Determine $\lim_{x\to\infty}\frac{1}{x}$ and $\lim_{x\to-\infty}\frac{1}{x}$.

Theorem: If $r > 0$ is a rational number, then $\lim_{x\to\infty}\frac{1}{x^r} = 0$. Also $\lim_{x\to-\infty}\frac{1}{x^r} = 0$ if x^r is defined for all *x*.

Example 4: Determine $\lim_{x\to\infty}\left(9+\frac{3}{x^4}\right)$

To take advantage of this theorem when determining the limit at infinity of a quotient, divide numerator and denominator by the largest power of the variable in the denominator.

Example 5: Find $\lim_{x\to\infty}\frac{3-2x}{4x+6}$.

Example 6: Find $\lim\limits_{x\to\infty}\dfrac{2x^2+15x+9}{5x^3-14}$ and $\lim\limits_{x\to-\infty}\dfrac{2x^2+15x+9}{5x^3-14}$.

Example 7: Find the horizontal asymptote (if any) of $h(x)=\dfrac{8x^2-6x+1}{3x^2+4x-5}$.

Example 8: Determine $\lim\limits_{x\to\infty}\dfrac{7x^5-5x+1}{8-x^2}$ and $\lim\limits_{x\to-\infty}\dfrac{7x^5-5x+1}{8-x^2}$.

Example 9: Determine $\lim_{x \to \infty} \frac{\sqrt[3]{x} - \sqrt{x}}{8 - x^{-2}}$

Example 10: Evaluate the limit of $f(x) = \frac{x^3 + 4x^2 - 11x - 7}{x^2 + 6x + 1}$ as x approaches $\pm\infty$.

Example 11: Evaluate the limit of $f(x) = \frac{8x^2 - 7x + 1}{2x - 3}$ as x approaches $\pm\infty$.

Slant asymptotes:

<u>Note</u>:
The graphs of the functions in the previous two examples have *oblique (slant) asymptotes*. This is because the function values (y-values) approached those of a linear function $y = mx + b$ as x approached $\pm\infty$.

<u>Example 12:</u> Evaluate the limit of $f(x) = \dfrac{2x^3 - x^2 + x}{x - 3}$ as x approaches $\pm\infty$. Does the graph of this function have a slant asymptote?