

3.5: Limits at Infinity

There are two types of limits involving infinity.

Limits at infinity, written in the form $\lim_{x \rightarrow \infty} f(x)$ or $\lim_{x \rightarrow -\infty} f(x)$, are related to horizontal asymptotes.

Infinite limits (covered in Section 1.5) take the form of statements like $\lim_{x \rightarrow a} f(x) = \infty$ or $\lim_{x \rightarrow a} f(x) = -\infty$. Infinite limits generally result in vertical asymptotes.

When combined, these two types of limits involving infinity result in statements such as $\lim_{x \rightarrow \infty} f(x) = \infty$ or $\lim_{x \rightarrow -\infty} f(x) = -\infty$, which describe the end behavior of graphs.

Limits at infinity:

Let f be a function defined on some interval (a, ∞) . Then

$$\lim_{x \rightarrow \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

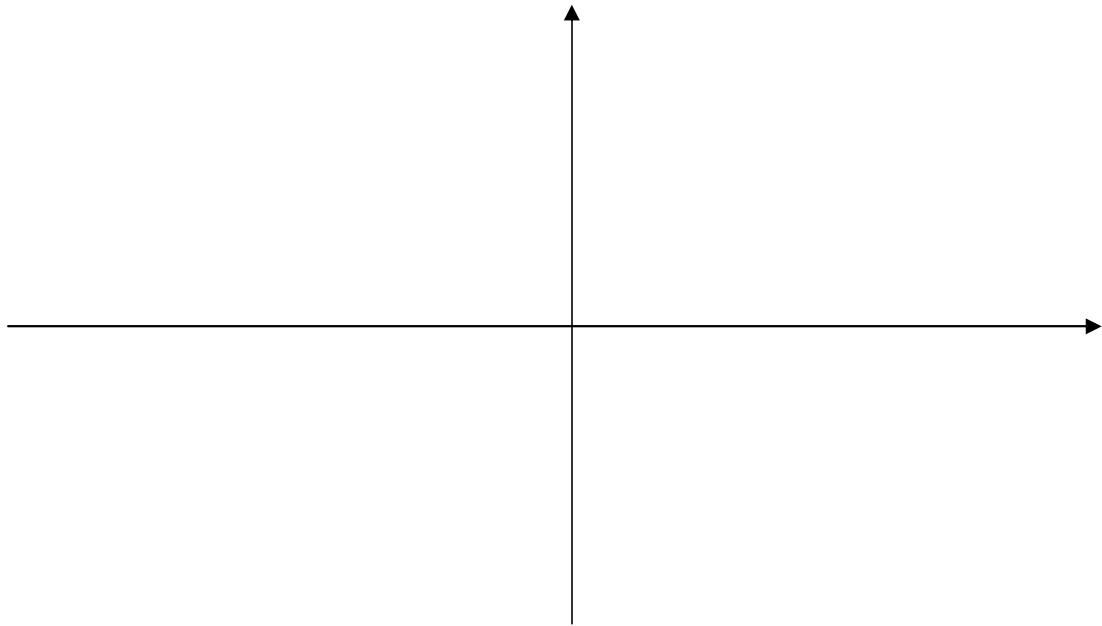
More precisely, $\lim_{x \rightarrow \infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number $M > 0$ such that for all x , $|f(x) - L| < \varepsilon$ whenever $x > M$.

Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \rightarrow -\infty} f(x) = L$$

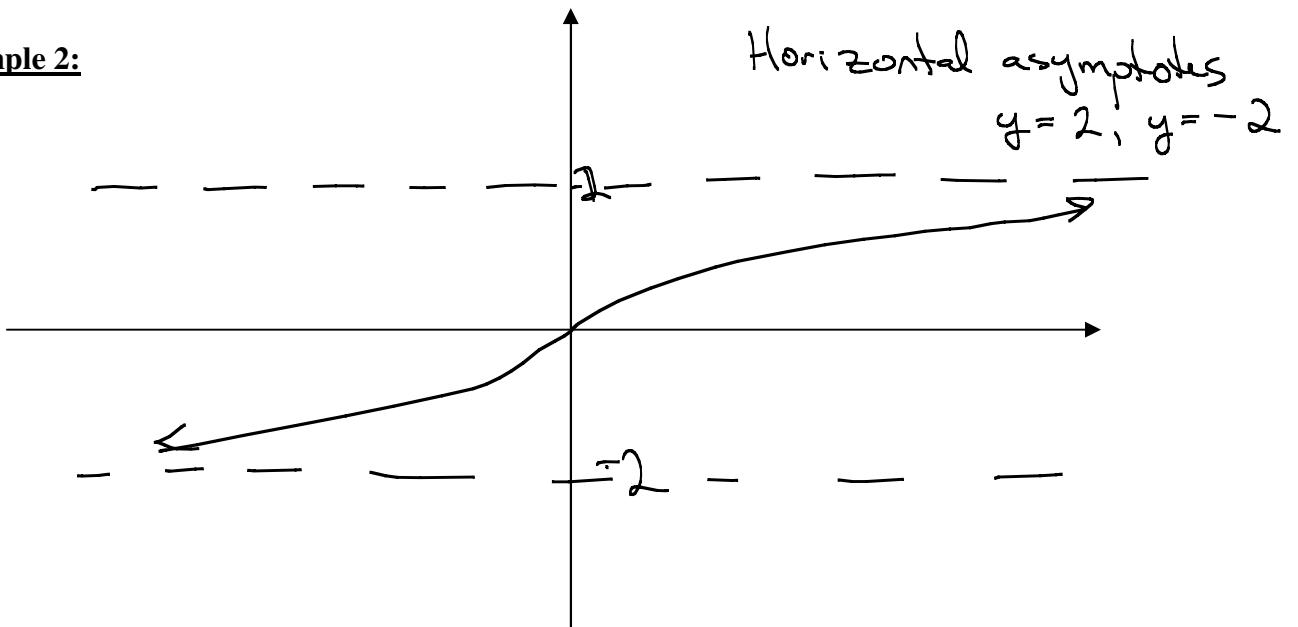
means that the values of $f(x)$ can be made arbitrarily close to L by making x a sufficiently large negative number.

More precisely, $\lim_{x \rightarrow -\infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number $N < 0$ such that for all x , $|f(x) - L| < \varepsilon$ whenever $x < N$.

Example 1:**Horizontal asymptotes:**

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = L \text{ or } \lim_{x \rightarrow -\infty} f(x) = L.$$

Example 2:

Example 3: Determine $\lim_{x \rightarrow \infty} \frac{1}{x}$ and $\lim_{x \rightarrow -\infty} \frac{1}{x}$.

As $x \rightarrow +\infty$, $\frac{1}{x} \rightarrow \frac{1}{\text{+huge}} \rightarrow \text{+tiny} \rightarrow 0$

As $x \rightarrow -\infty$, $\frac{1}{x} \rightarrow \frac{1}{-\text{huge}} \rightarrow \text{-tiny} \rightarrow 0$

$$\text{So } \lim_{x \rightarrow \infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

could write $\lim_{x \rightarrow \pm\infty} \frac{1}{x} = 0$

Theorem: If $r > 0$ is a rational number, then $\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$. Also $\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$ if x^r is defined for all x .

Example 4: Determine $\lim_{x \rightarrow \infty} \left(9 + \frac{3}{x^4} \right)$

$$\lim_{x \rightarrow \infty} \left(9 + \frac{3}{x^4} \right) = 9 + 0 = \boxed{9}$$

To take advantage of this theorem when determining the limit at infinity of a quotient, divide numerator and denominator by the largest power of the variable in the denominator.

Example 5: Find $\lim_{x \rightarrow \infty} \frac{3-2x}{4x+6}$.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3-2x}{4x+6} &= \lim_{x \rightarrow \infty} \frac{\frac{3}{x} - \frac{2x}{x}}{\frac{4x}{x} + \frac{6}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{3}{x} - 2}{4 + \frac{6}{x}} \\ &= \frac{0-2}{4+0} = \frac{-2}{4} = \boxed{-\frac{1}{2}} \end{aligned}$$

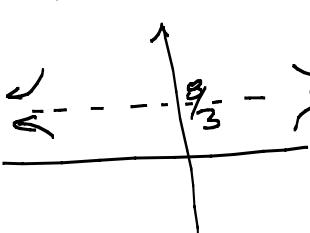
Example 6: Find $\lim_{x \rightarrow \infty} \frac{2x^2 + 15x + 9}{5x^3 - 14}$ and $\lim_{x \rightarrow -\infty} \frac{2x^2 + 15x + 9}{5x^3 - 14}$.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2x^2 + 15x + 9}{5x^3 - 14} &= \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^3} + \frac{15x}{x^3} + \frac{9}{x^3}}{\frac{5x^3}{x^3} - \frac{14}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{2}{x} + \frac{15}{x^2} + \frac{9}{x^3}}{5 - \frac{14}{x^3}} = \frac{0 + 0 + 0}{5 - 0} = \frac{0}{5} = 0 \quad \boxed{0} \\ \lim_{x \rightarrow -\infty} \frac{2x^2 + 15x + 9}{5x^3 - 14} &= \lim_{x \rightarrow -\infty} \frac{\frac{2}{x} + \frac{15}{x^2} + \frac{9}{x^3}}{5 - \frac{14}{x^3}} = \frac{0 + 0 + 0}{5 - 0} = \frac{0}{5} = 0 \quad \boxed{0} \end{aligned}$$

Example 7: Find the horizontal asymptote (if any) of $h(x) = \frac{8x^2 - 6x + 1}{3x^2 + 4x - 5}$.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{8x^2 - 6x + 1}{3x^2 + 4x - 5} &= \lim_{x \rightarrow \infty} \frac{\frac{8x^2}{x^2} - \frac{6x}{x^2} + \frac{1}{x^2}}{\frac{3x^2}{x^2} + \frac{4x}{x^2} - \frac{5}{x^2}} = \lim_{x \rightarrow \infty} \frac{8 - \frac{6}{x} + \frac{1}{x^2}}{3 + \frac{4}{x} - \frac{5}{x^2}} \\ &= \frac{8 - 0 + 0}{3 + 0 - 0} = \frac{8}{3} \end{aligned}$$

Horizontal asymptote:
 $y = \frac{8}{3}$



Example 8: Determine $\lim_{x \rightarrow \infty} \frac{7x^5 - 5x + 1}{8 - x^2}$ and $\lim_{x \rightarrow -\infty} \frac{7x^5 - 5x + 1}{8 - x^2}$.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{7x^5 - 5x + 1}{8 - x^2} &= \lim_{x \rightarrow \infty} \frac{\frac{7x^5}{x^2} - \frac{5x}{x^2} + \frac{1}{x^2}}{\frac{8}{x^2} - \frac{x^2}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{7x^3}{x} - \frac{5}{x} + \frac{1}{x^2}}{\frac{8}{x^2} - 1} = \frac{\lim_{x \rightarrow \infty} (7x^3) - 0 + 0}{0 - 1} \\ &\stackrel{\substack{1 (+\text{huge})^3 \\ -1 \\ \rightarrow \frac{+\text{huge}}{-1} \rightarrow -\text{huge}}}{=} \frac{\lim_{x \rightarrow \infty} (7x^3)}{-1} = \boxed{-\infty} \end{aligned}$$

Note: $\infty - \infty$ is an indeterminate form.

(as are $\frac{0}{0}$, $\frac{\infty}{\infty}$)

$$\begin{aligned} \text{Note: } \frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} &= x^{\frac{1}{2} - \frac{1}{3}} \\ &= x^{\frac{3}{6} - \frac{2}{6}} = x^{\frac{1}{6}} \end{aligned}$$

Example 9: Determine $\lim_{x \rightarrow \infty} \frac{\sqrt[3]{x} - \sqrt{x}}{8 - x^{-2}}$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\sqrt[3]{x} - \sqrt{x}}{8 - \frac{1}{x^2}} &= \lim_{x \rightarrow \infty} \frac{x^{\frac{1}{3}} - x^{\frac{1}{2}}}{8 - \frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{x^{\frac{1}{3}}(1 - x^{\frac{1}{6}})}{8 - \frac{1}{x^2}} = \boxed{-\infty} \end{aligned}$$

+huge(1-huge) \rightarrow +huge(-huge) = -huge

Example 10: Evaluate the limit of $f(x) = \frac{x^3 + 4x^2 - 11x - 7}{x^2 + 6x + 1}$ as x approaches $\pm\infty$.

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^3 + 4x^2 - \frac{11}{x} - \frac{1}{x^2}}{1 + \frac{6}{x} + \frac{1}{x^2}} = \frac{\lim_{x \rightarrow \infty} x^3 + 4x^2 - 0 + 0}{1 + 0 + 0}$$

As $x \rightarrow \infty$,
+huge +4
 \rightarrow +huge

$$= \frac{\lim_{x \rightarrow \infty} x^3 + 4x^2}{1} = \boxed{+\infty}$$

$$\lim_{x \rightarrow -\infty} f(x) = \frac{\lim_{x \rightarrow -\infty} x^3 + 4x^2}{1} = \boxed{-\infty}$$

Example 11: Evaluate the limit of $f(x) = \frac{8x^2 - 7x + 1}{2x - 3}$ as x approaches $\pm\infty$.

Find the slant asymptote.

$$\begin{array}{r} \text{slant} \\ \hline 2x - 3 \overline{) 8x^2 - 7x + 1} \\ \underline{- (8x^2 - 12x)} \\ \quad \quad \quad 5x + 1 \\ \quad \quad \quad - (5x - \frac{15}{2}) \\ \hline \quad \quad \quad 1 + \frac{17}{2} = \frac{17}{2} \end{array}$$

$$\lim_{x \rightarrow +\infty} f(x) = +\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

So the slant asymptote is $y = 4x + \frac{5}{2}$

$$f(x) = 4x + \frac{5}{2} + \frac{\frac{17}{2}}{2x - 3}$$

so as $x \rightarrow \pm\infty$,
 $y \rightarrow 4x + \frac{5}{2} + 0$

Slant asymptotes:

Note:

The graphs of the functions in the previous two examples have *oblique (slant) asymptotes*.

This is because the function values (y -values) approached those of a linear function

$y = mx + b$ as x approached $\pm\infty$.

Example 12: Evaluate the limit of $f(x) = \frac{2x^3 - x^2 + x}{x-3}$ as x approaches $\pm\infty$. Does the graph of this function have a slant asymptote?

$$\begin{array}{r}
 2x^2 + 5x + 16 \\
 x - 3 \overline{)2x^3 - x^2 + x + 0} \\
 \underline{- (2x^3 - 6x^2)} \\
 \quad \quad \quad 5x^2 + x \\
 \quad \quad \quad \underline{- (5x^2 - 15x)} \\
 \quad \quad \quad \quad \quad 16x + 0 \\
 \quad \quad \quad \quad \quad \underline{- (16x - 48)} \\
 \quad \quad \quad \quad \quad \quad \quad 48
 \end{array}$$

$$f(x) = 2x^2 + 5x + 16 + \frac{48}{x-3}$$

Graph it. parabola

$$y = 2x^2 + 5x + 16$$

$$y = 2\left(x^2 + \frac{5}{2}x + \frac{25}{16}\right) + 16 - \frac{25}{8}$$

$$\left(\frac{(5)}{2}\right)\left(\frac{1}{2}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{16}$$

$$y = 2\left(x + \frac{5}{4}\right)^2 + \frac{103}{8}$$

Problem #8 cont'd

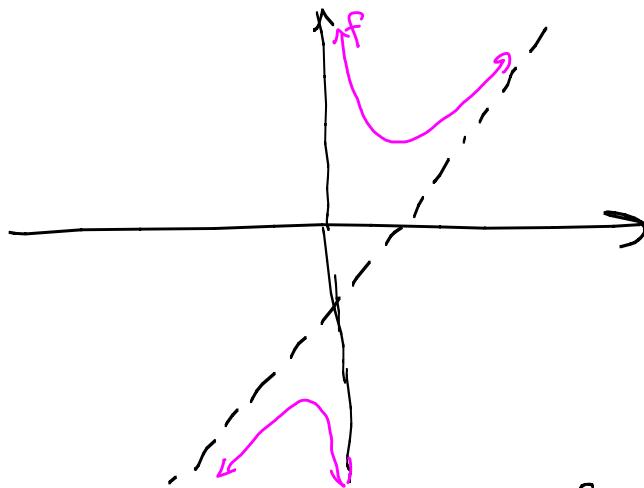
$$\lim_{x \rightarrow -\infty} \left(\frac{7x^5 - 5x + 1}{8 - x^2} \right) = \lim_{x \rightarrow -\infty} \left(\frac{7x^3 - \frac{5}{x} + \frac{1}{x^2}}{\frac{8}{x^2} - 1} \right)$$

$$= \frac{\lim_{x \rightarrow -\infty} (7x^3)}{0 - 1} - 0 + 0$$

$$= \frac{\lim_{x \rightarrow -\infty} (7x^3)}{-1} = \boxed{+\infty}$$

$(-huge)^3$
-1
-huge
-1
+ huge

Slant asymptote: occurs when $\deg(\text{denom}) = \deg(\text{num}) - 1$



$$\lim_{x \rightarrow \infty} f(x) = +\infty$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

Example 10 revisited:

$$\lim_{x \rightarrow \pm\infty} \frac{x^3 + 4x^2 - 11x - 7}{x^2 + 6x + 1}$$

$$\begin{array}{r} x-2 \\ \hline x^2 + 6x + 1 \sqrt{x^3 + 4x^2 - 11x - 7} \\ - (x^3 + 6x^2 + x) \\ \hline -2x^2 - 12x - 7 \\ - (-2x^2 - 12x - 2) \\ \hline -5 \end{array}$$

$$f(x) = \frac{x^3 + 4x^2 - 11x - 7}{x^2 + 6x + 1} = x - 2 - \frac{5}{x^2 + 6x + 1}$$

As $x \rightarrow \pm\infty$, $y \rightarrow x - 2 - 0$

so $y = x - 2$ is the slant asymptote