

3.6: A Summary of Curve Sketching

Steps for Curve Sketching

1. Determine the domain of f .
2. Find the x -intercepts and y -intercept, if any.
3. Determine the “end behavior” of f , that is, the behavior for large values of $|x|$ (limits at infinity). (usually we can get this from increasing/decreasing intervals)
4. Find the vertical, horizontal, and oblique asymptotes, if any.
5. Determine the intervals where f is increasing/decreasing.
6. Find the relative extremes of f , if any. (You should find both the x - and y -values.)
7. Determine the intervals where f is concave up/concave down.
8. Find the inflection points, if any. (You should find both the x - and y -values.)
9. Plot more points if necessary, and sketch the graph.

Example 1: Sketch the graph of $f(x) = x^3 - 6x^2 + 9x$.

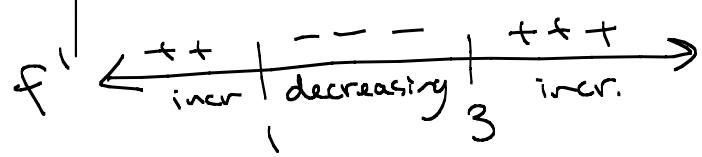
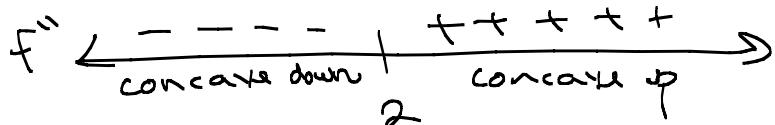
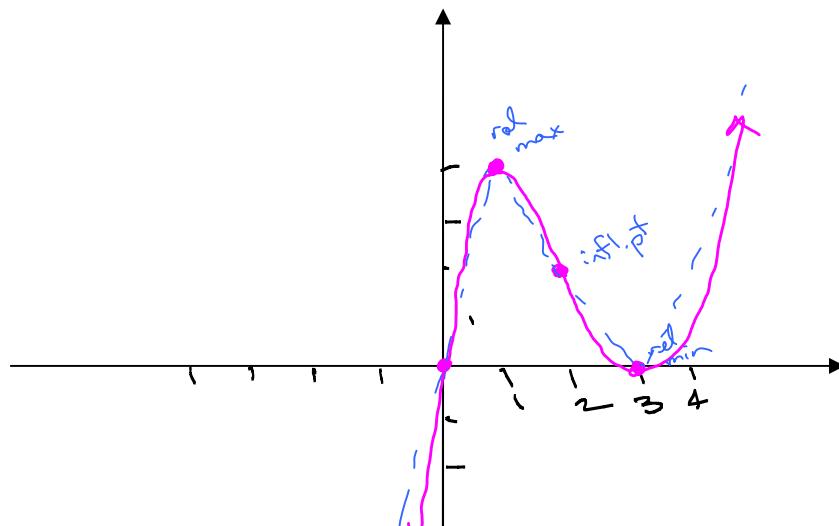
Domain: $(-\infty, \infty)$

Find intercepts: $f(x) = x(x^2 - 6x + 9) = x(x-3)^2$

Set $y=0$: $0 = x(x-3)^2$
 $x=0, x=3$

Set $x=0$: $f(x) = x^3 - 6x^2 + 9x$
 $f(0) = 0^3 - 6(0)^2 + 9(0) = 0$ so y -int is 0.

x-intercepts: 0, 3
 or $(0,0), (3,0)$
 y-intercept: 0
 or $(0,0)$
 Relative max at $(1,4)$.
 Relative min at $(3,0)$.
 No asymptotes
 Inflection pt $(2,2)$



Example 2: Sketch the graph of $f(x) = 3x^4 + 4x^3$.

$$f(x) = x^3(3x+4)$$

Find x-intercepts: Set $f(x) = 0$: $0 = x^3(3x+4)$

$$x = 0, -\frac{4}{3}$$

Find y-intercept: Set $x=0$: $y = 3(0)^4 + 4(0)^3$

$$y = 0$$

Find increasing/decreasing intervals:

$$\begin{aligned} f'(x) &= 12x^3 + 12x^2 \\ &= 12x^2(x+1) \end{aligned}$$

Setting $f'(x) = 0$ gives critical #s $0, -1$.

$$\begin{aligned} (-\infty, -1): \text{ Test } x &= -2 \\ f'(-2) &= 12(-2)^2(-2+1) \\ &\Rightarrow (+)(+)(-) \\ &\Rightarrow (-) \end{aligned}$$

$$\begin{aligned} (-1, 0): \text{ Test } x &= -0.5 \\ f'(-0.5) &= 12(-0.5)^2(-0.5+1) \\ &\Rightarrow (+)(+)(+) \end{aligned}$$

$$\begin{aligned} (0, \infty): \text{ Test } x &= 1 \\ f'(1) &= 12(1)^2(1+1) \\ &\quad (+) \end{aligned}$$

$$\begin{aligned} \text{Relative min at } x &= -1 \\ \text{Find } y: f(-1) &= 3(-1)^4 + 4(-1)^3 \\ &= 3 - 4 = -1 \end{aligned}$$

Concavity cont'd

$$\left(-\frac{2}{3}, 0\right): \text{ Test } x = -\frac{2}{3}$$

$$\begin{aligned} f''\left(-\frac{2}{3}\right) &= 12\left(-\frac{2}{3}\right)\left(3\left(-\frac{2}{3}\right)^2\right), \\ &= (+)(-)(+) \\ &\Rightarrow (-) \end{aligned}$$

Inflection pt at $x = -\frac{2}{3}, 0$

Find y-values:

$$\begin{aligned} f(0) &= 0 \\ f\left(-\frac{2}{3}\right) &= 3\left(-\frac{2}{3}\right)^4 + 4\left(-\frac{2}{3}\right)^3 \\ &= 3\left(\frac{16}{81}\right) + 4\left(-\frac{8}{27}\right) \\ &= \frac{48}{27} - \frac{32}{27} = -\frac{16}{27} \end{aligned}$$

x-intercepts: $0, -\frac{4}{3}$
or $(0, 0), \left(-\frac{4}{3}, 0\right)$

y-intercept: 0 or $(0, 0)$

No asymptotes

$$\begin{array}{c|ccc} f' & \text{---} & ++ & ++ \\ \text{decr.} & | & \text{incr} & \text{incr} \\ \hline & 0 & & \end{array}$$

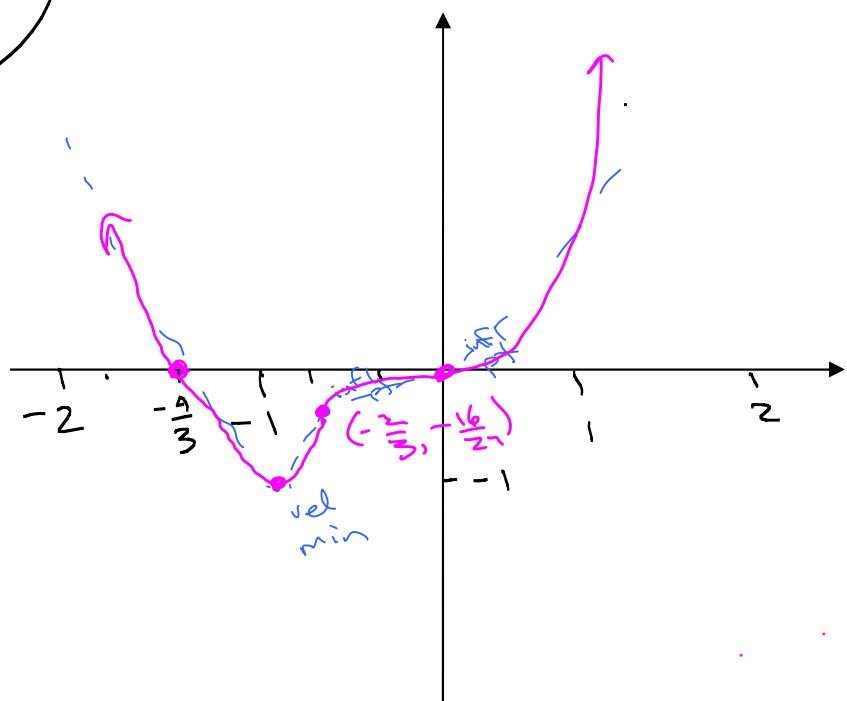
Relative min: $(-1, -1)$

$$\begin{array}{c|ccc} f'' & ++ & - & ++ \\ \text{conc up} & | & \text{conc down} & \text{conc up} \\ \hline -\frac{2}{3} & 0 & & \end{array}$$

$$\begin{aligned} \text{Find concavity: } f''(x) &= 36x^2 + 24x \\ &= 12x(3x+2) \end{aligned}$$

$$f''(x) = 0 \text{ for } x = 0, x = -\frac{2}{3}$$

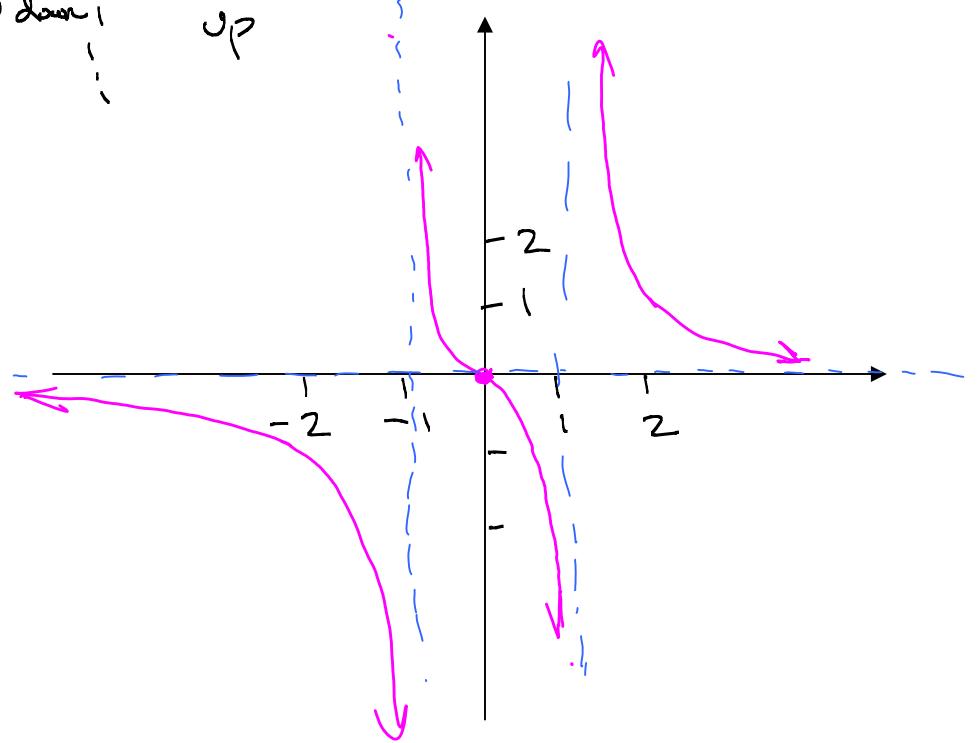
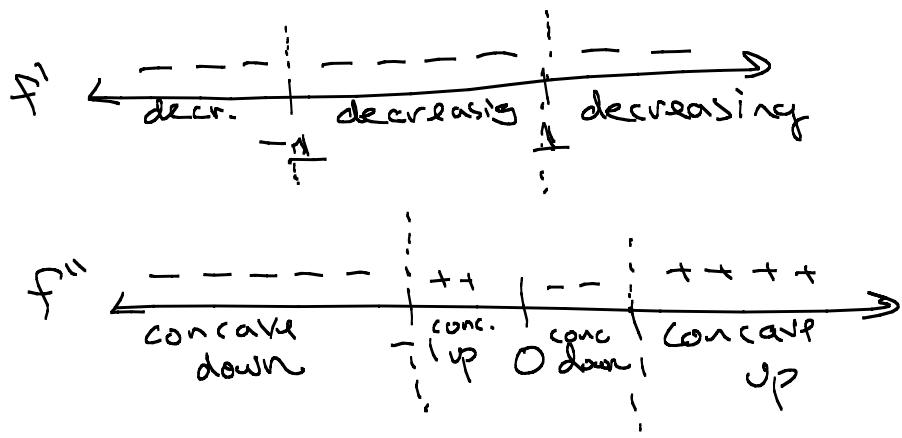
$$\begin{aligned} (-\infty, -\frac{2}{3}): \text{ Test } x &= -1 \\ f''(-1) &= 12(-1)(3(-1)+2) \Rightarrow (+)(-) \Rightarrow (+) \end{aligned}$$



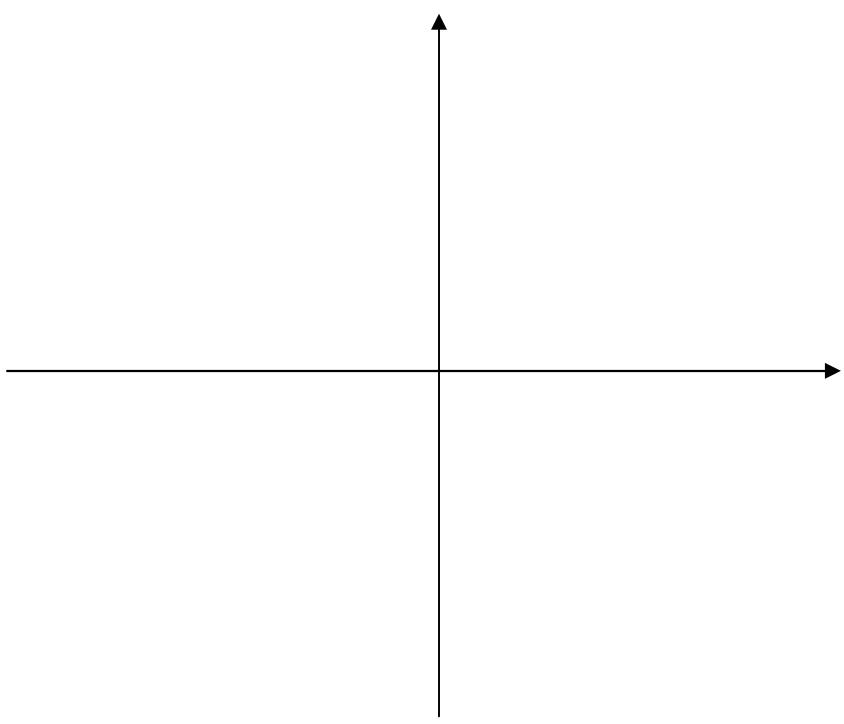
Inflection pts: $(0, 0)$ and $\left(-\frac{2}{3}, -\frac{16}{27}\right)$

Example 3: Sketch the graph of $f(x) = \frac{2x}{x^2 - 1}$.

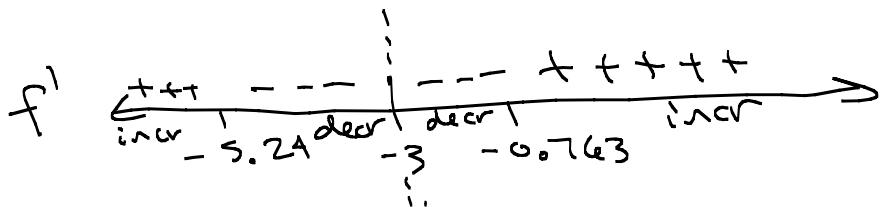
$$f(x) = \frac{2x}{(x+1)(x-1)}$$



Example 4: Sketch the graph of $f(x) = \frac{x^2 + 1}{x^2 - 4}$.



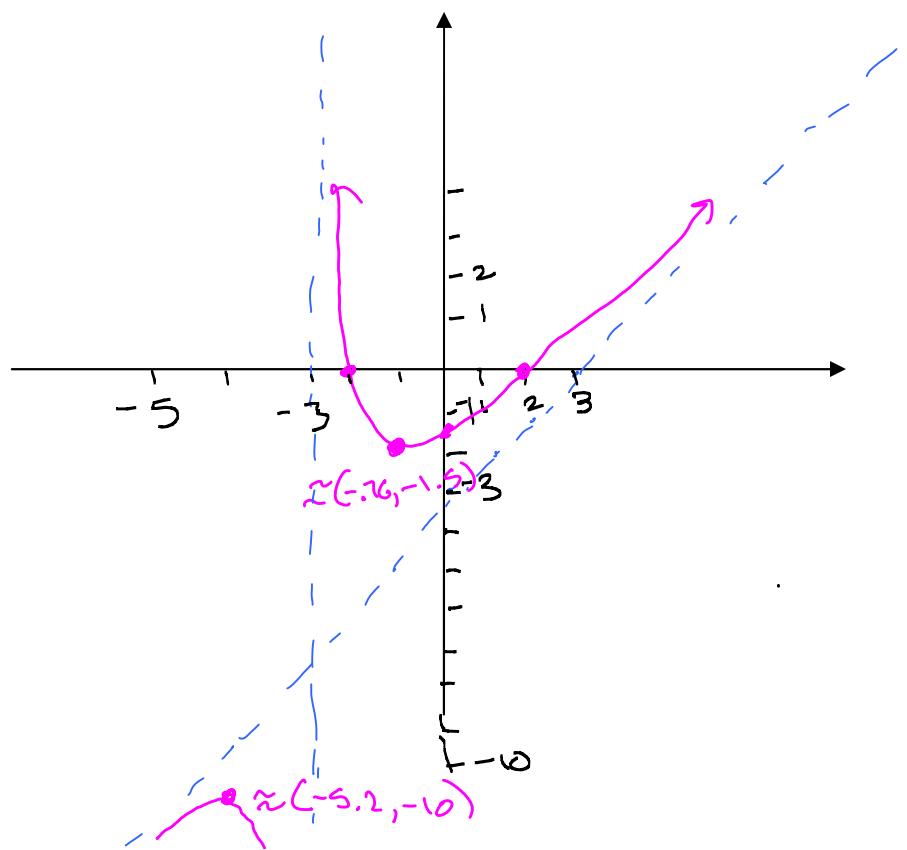
Example 5: Sketch the graph of $f(x) = \frac{x^2 - 4}{x + 3}$.



Oblique asymptote: $y = x - 3$

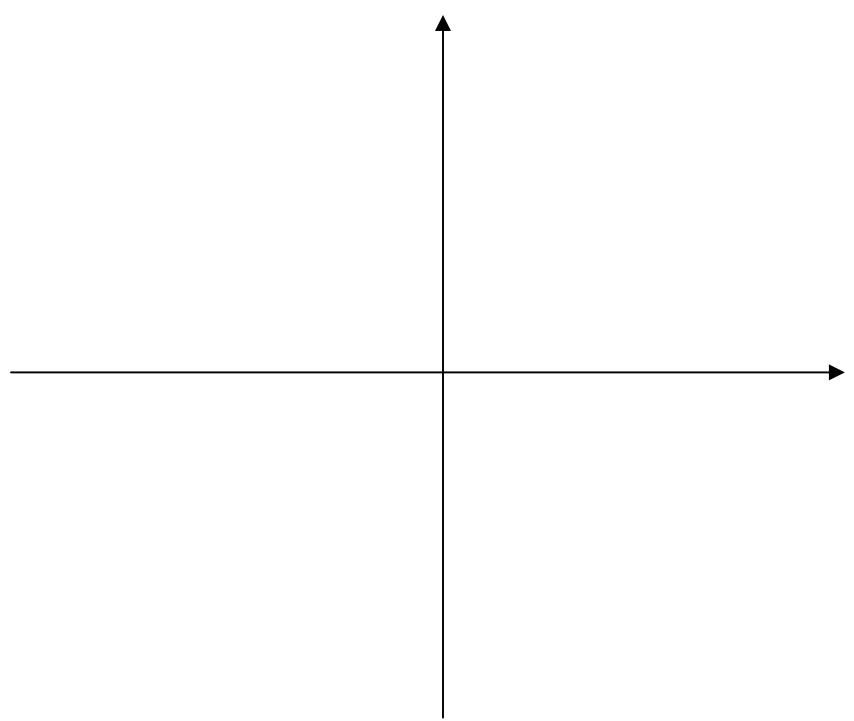
Relative min $\approx f(-0.763) \approx -1.5$

Relative max $\approx f(-5.24) \approx -10.$



3.6.6

Example 6: Sketch the graph of $f(x) = 5x^{\frac{2}{3}} - x^{\frac{5}{3}}$.



Example 7: Sketch the graph of $f(x) = x + \cos x$ on the interval $[-2\pi, 2\pi]$.

Find x -intercepts: Set $y=0$: $0 = x + \cos x$

$$-x = \cos x$$

can't solve
algebraically

Find y -intercept: Set $x=0$

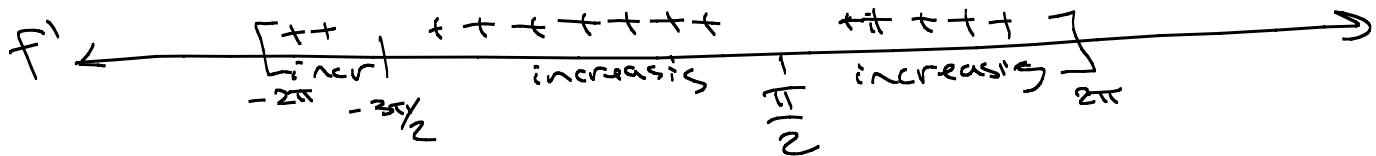
$$y = 0 + \cos 0 = 0 + 1 = 1$$

$$f'(x) = 1 - \sin x$$

$$\text{Set } f'(x)=0: 0 = 1 - \sin x$$

$$\sin x = 1$$

$$x = \frac{\pi}{2}, -\frac{3\pi}{2} \text{ critical HS.}$$



$$(-2\pi, -\frac{3\pi}{2}): \text{Test } x = -\frac{11\pi}{6} \Rightarrow f'(-\frac{11\pi}{6}) = 1 - \sin(-\frac{11\pi}{6}) = 1 - \frac{1}{2} = \frac{1}{2} (+)$$

$$(-\frac{3\pi}{2}, \frac{\pi}{2}): \text{Test } x = 0 \Rightarrow 1 - \sin 0 = 1 - 0 = 1 (+)$$

$$(\frac{\pi}{2}, 2\pi): \text{Test } x = \pi \Rightarrow 1 - \sin \pi = 1 - 0 = 1 (+)$$

f is increasing on $(-2\pi, 2\pi)$

Find y -value at endpoints:

$$f(2\pi) = 2\pi + \cos 2\pi$$

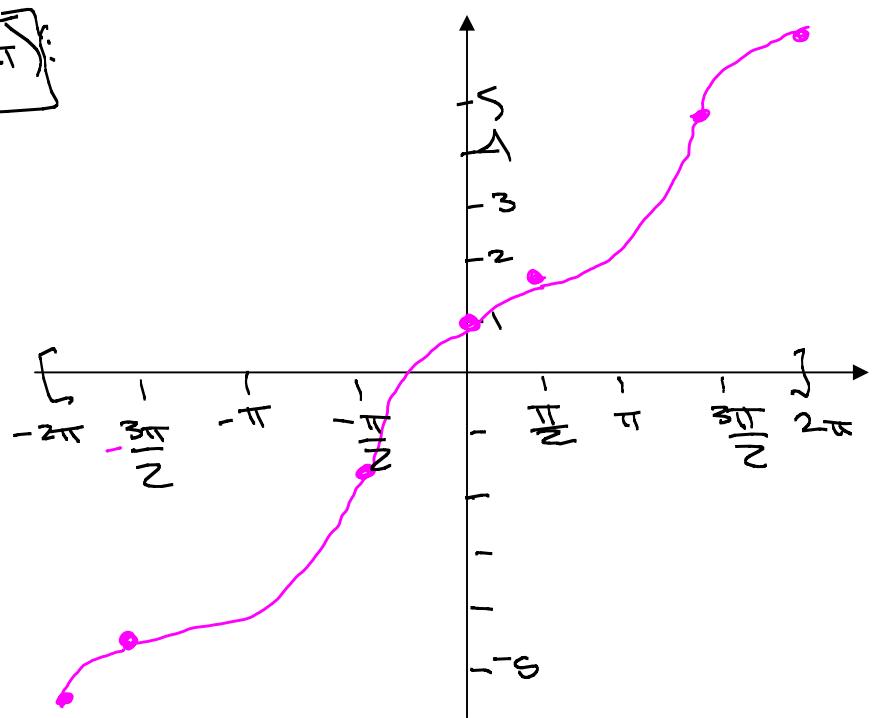
$$= 2\pi + 1$$

$$\approx 6.28 + 1$$

$$\approx 7.28$$

$$f(-2\pi) = -2\pi + \cos(-2\pi)$$

$$\approx -5.28$$



Ex 1 cont'd:

$$f(x) = x^3 - 6x^2 + 9x$$

$$f'(x) = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x-3)(x-1)$$

Find critical #s: $0 = 3(x-3)(x-1)$
 $x=3, x=1$ critical #s

Test for increasing/decreasing:

$(-\infty, 1)$: Test $x=0$

$$f'(0) = 3(0)^2 - 12(0) + 9 = +9 (+)$$

$(1, 3)$: Test $x=1.5$

$$f'(x) = 3(x-3)(x-1)$$

$$f'(1.5) = 3(1.5-3)(1.5-1)$$

$$\Rightarrow (+)(-) (+)$$

$$\Rightarrow (-)$$

$(3, \infty)$: Test $x=4$

$$f'(x) = 3(4-3)(4-1)$$

$$\Rightarrow (+)(+) (+)$$

$$\Rightarrow (+)$$

From sign chart for f' ,

we have a local max at 1
and local min at 3

Need the y-values: $f(x) = x^3 - 6x^2 + 9x$

$$f(1) = 1^3 - 6(1)^2 + 9(1) = 1 - 6 + 9$$

$$= -5 + 9 = 4$$

$$f(3) = (3)^3 - 6(3)^2 + 9(3)$$

$$= 27 - 54 + 27 = 0$$

Find concavity:

$$f'(x) = 3x^2 - 12x + 9$$

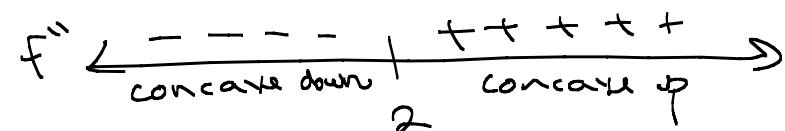
$$f''(x) = 6x - 12 = 6(x-2)$$

Set $f''(x) = 0$: $0 = 6(x-2)$.

$$f''(x) = 0 \text{ for } x=2$$

$(-\infty, 2)$: Test $x=0$

$$f''(0) = 6(0)-12 = -12 (-)$$



inflection pt at $x=2$.

Find the y-value:

$$f(2) = 2^3 - 6(2)^2 + 9(2)$$

$$= 8 - 24 + 18$$

$$= -16 + 18 = 2$$

Example #3 $f(x) = \frac{2x}{x^2 - 1}$

$$\begin{aligned}
 f'(x) &= \frac{(x^2-1)(2) - 2x(2x)}{(x^2-1)^2} = \frac{2x^2 - 2 - 4x^2}{(x^2-1)^2} = \frac{-2x^2 - 2}{(x^2-1)^2} = \frac{-2(x^2+1)}{(x^2-1)^2} \\
 f''(x) &= \frac{d}{dx} \left(\frac{-2x^2-2}{(x^2-1)^2} \right) = \frac{(x^2-1)^2 \frac{d}{dx}(-2x^2-2) - (-2x^2-2) \frac{d}{dx}(x^2-1)^2}{[(x^2-1)^2]^2} \\
 &= \frac{(x^2-1)^2(-4x) + (2x^2+2)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{-4x(x^2-1)^2 + 2x(2x^2+2)(x^2-1)}{(x^2-1)^4} \\
 &= \frac{4x(x^2-1)[-1(x^2-1) + 2x^2+2]}{(x^2-1)^4} = \frac{4x[-x^2+1 + 2x^2+2]}{(x^2-1)^3} \\
 &= \frac{4x(x^2+3)}{(x^2-1)^3}
 \end{aligned}$$

From original function

$$f(x) = \frac{2x}{x^2-1} = \frac{2x}{(x+1)(x-1)}$$

Vertical asymptotes $x = 1$
 $x = -1$
 (comes from setting denominator = 0)

x -intercept: 0 or $(0, 0)$
 (comes from setting numerator = 0)

Find y-intercept: set $x = 0$.

$$y = f(0) = \frac{2(0)}{0^2-1} = \frac{0}{-1} = 0 \Rightarrow y\text{-intercept is } 0 \text{ or } (0, 0).$$

Find horizontal asymptotes

$\deg(\text{denom}) > \deg(\text{num})$, so horizontal asymptote is $y = 0$.

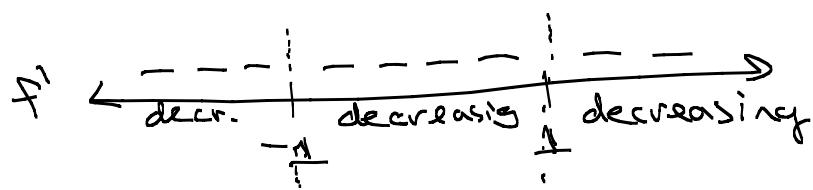
$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \frac{2x}{x^2-1} = \lim_{x \rightarrow \pm\infty} \frac{\frac{2}{x}}{1 - \frac{1}{x^2}} = \frac{0}{1-0} = 0$$

From 1st derivative
 $f'(x) = \frac{-2(x^2+1)}{(x^2-1)^2}$

f' is undefined at $x = \pm 1$. f is also undefined at $x = \pm 1$.

Where is $f'(x) = 0$?

Nowhere. So no critical pts



$$\frac{-2(x^2+1)}{(x^2-1)^2} = 0 \quad (\text{setting } f'(x)=0)$$

$$-2(x^2+1) = 0$$

$$x^2+1=0 \\ x^2=-1 \quad \text{no real solutions.}$$

Note: $f'(x) = \frac{-2(x^2+1)}{(x^2-1)^2} \Rightarrow \frac{(-)(+)}{(+)} \Rightarrow (-) \text{ for all } x$

No relative extrema.

From 2nd derivative:

$$f''(x) = \frac{4x(x^2+3)}{(x^2-1)^3}$$

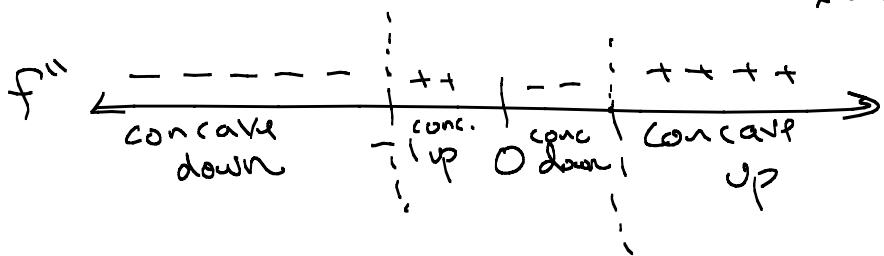
Where is $f''(x)$ undefined?

At $x=\pm 1$, same as f

Where is $f''(x) = 0$? At $x=0$

$$0 = 4x(x^2+3)$$

$$\begin{array}{l|l} 4x=0 & x^2+3=0 \\ x=0 & x^2=-3 \end{array} \quad \text{no real solns}$$



$(-\infty, -1)$: Test $x=-2$

$$f''(-2) \Rightarrow \frac{4(-2)(+)}{(-2^2-1)^3}$$

$$\Rightarrow \frac{(-)(+)}{(-1)^3} \Rightarrow \frac{(-)}{(+)^3} \Rightarrow (-)$$

$(-1, 0)$: Test $x=-0.5$

$$f''(-0.5) \Rightarrow \frac{4(-0.5)(+)}{((-0.5)^2-1)^3} \Rightarrow \frac{(-)(+)}{(-)^3} \Rightarrow (+)$$

$(0, 1)$: Test $x=0.5$

$$f''(0.5) \Rightarrow \frac{4(0.5)(+)}{(0.5^2-1)^3} \Rightarrow \frac{(+)}{(-)^3} \Rightarrow \frac{(+)}{(-)} \Rightarrow (-)$$

$(1, \infty)$: $f''(x) > 0$

Inflection point at $x=0$: Find y -value:

$$f(0) = 0 \quad \text{Inflection pt: } (0, 0)$$

$$\text{Ex 5: } f(x) = \frac{x^2 - 4}{x + 3} = \frac{(x+2)(x-2)}{x+3}$$

From original function:

Vertical Asymptote $x = -3$

x -intercepts: 2 and -2

(2,0) and (-2,0)

y -intercept: $-\frac{4}{3}$

Find the oblique asymptote:

$$\begin{array}{r} x-3 \\ x+3 \overline{)x^2 + 0x - 4} \\ - (x^2 + 3x) \\ \hline -3x - 4 \\ - (-3x - 9) \\ \hline 5 \end{array}$$

$$f(x) = x - 3 + \frac{5}{x+3}$$

Oblique asymptote:

$$y = x - 3$$

$$f(x) = \frac{x^2 - 4}{x + 3}$$

$$f'(x) = \frac{(x+3)(2x) - (x^2 - 4)(1)}{(x+3)^2} = \frac{2x^2 + 6x - x^2 + 4}{(x+3)^2} = \frac{x^2 + 6x + 4}{(x+3)^2}$$

$f'(x)$ is undefined at $x = -3$.

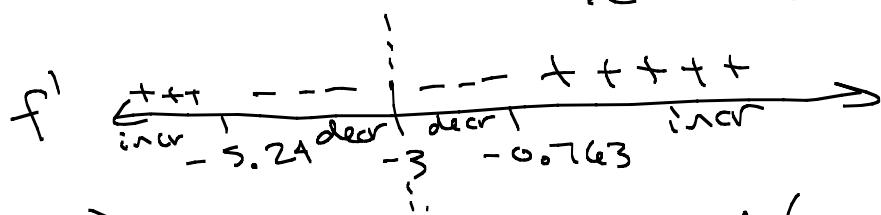
Set numerator = 0 to find critical #s.

$$x^2 + 6x + 4 = 0$$

$$x = \frac{-6 \pm \sqrt{36 - 4(1)(4)}}{2(1)} = \frac{-6 \pm \sqrt{36 - 16}}{2} = \frac{-6 \pm \sqrt{20}}{2}$$

$$= \frac{-6 \pm 2\sqrt{5}}{2} = \frac{2(-3 \pm \sqrt{5})}{2} = -3 \pm \sqrt{5}$$

$$\approx -0.763, -5.236$$



$$(-\infty, -5.24);$$

Test number $x = -6$

$$f'(-6) = \frac{(-6)^2 + 6(-6) + 4}{(-6+3)} = \frac{0+4}{(-3)} = (+)$$

$$(-5.24, -3); \text{ Test } -4$$

$$f'(-4) = \frac{(-4)^2 + 6(-4) + 4}{(-4+3)} = (-)$$

$$\frac{16 - 24 + 4}{(-4+3)} = \frac{-4}{(-1)} = (+)$$

Find y-values: $f(x) = \frac{x^2 - 4}{x + 3}$

$$f(-3 + \sqrt{5}) \approx -1.527$$

$$f(-3 - \sqrt{5}) \approx -10.472$$

Ex. 7 cont'd:

2nd derivative:

$$f'(x) = 1 - \sin x$$

$$f''(x) = 0 - \cos x \\ = -\cos x$$

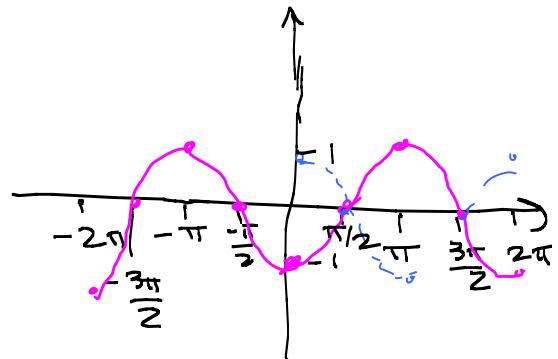
$$\text{Set } f''(x) = 0 : 0 = -\cos x \\ 0 = \cos x$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2}$$

$$\leftarrow \begin{array}{c|c|c|c|c|c|c} \text{---} & \text{+} & \text{---} & \text{+} & \text{---} & \text{+} & \text{---} \\ \text{down} & \uparrow & \text{up} & \downarrow & \text{down} & \uparrow & \text{up} \\ -2\pi & \frac{3\pi}{2} & -\frac{\pi}{2} & \frac{\pi}{2} & \frac{3\pi}{2} & \frac{\pi}{2} & 2\pi \end{array}$$

$$f''(x) = -\cos x$$

$$\text{Graph } y = -\cos x$$



Find y-values:

$$f\left(-\frac{3\pi}{2}\right) = -\frac{3\pi}{2} + \cos\left(-\frac{3\pi}{2}\right) = -\frac{3\pi}{2} + 0 = -\frac{3\pi}{2} \approx -4.7$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + \cos\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} \approx -1.57$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 0 \approx 1.57$$

$$f\left(\frac{3\pi}{2}\right) \approx 4.7$$

No Relative extrema

Inflection Points at $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$