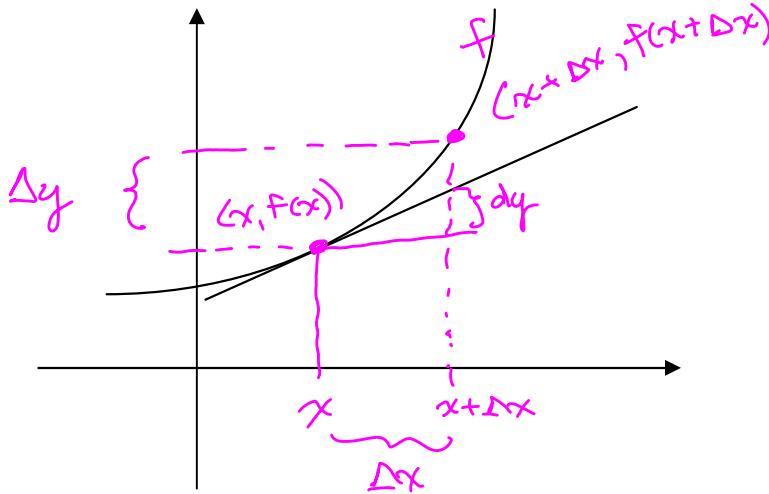


3.9: Differentials

Differentials:



$$\Delta y = f(x + \Delta x) - f(x)$$

$$\Delta x = dx, \text{ so}$$

$$\Delta y = f(x + dx) - f(x)$$

$$\text{Let } dx = \Delta x$$

If $y = f(x)$ is a differentiable function, we can let dx represent an amount of change in x .

Then the *differential* dy is defined to be $dy = f'(x)dx$. $\Rightarrow \frac{dy}{dx} = f'(x)$

dy is an approximation to $\Delta y = f(x + \Delta x) - f(x)$, which is the actual change in y .

Example 1: Find the differential dy for $y = \sqrt{6+x}$. Evaluate it when $x = 10$ and $dx = -0.3$.

Compare it to the exact value of Δy .

$$\begin{aligned} y &= \sqrt{6+x} \\ &= (6+x)^{1/2} \\ \frac{dy}{dx} &= \frac{1}{2}(6+x)^{-1/2}(1) \\ \frac{dy}{dx} &= \frac{1}{2\sqrt{6+x}} \end{aligned}$$

$$dy = \frac{dx}{2\sqrt{6+x}}$$

$$\begin{aligned} dy &= \frac{-0.3}{2\sqrt{6+10}} \\ &\quad \left| \begin{array}{l} x=10 \\ dx=-0.3 \end{array} \right. \\ &= \frac{-0.3}{2\cdot 4} \\ &= -\frac{0.3}{8} = \boxed{-0.0375} \end{aligned}$$

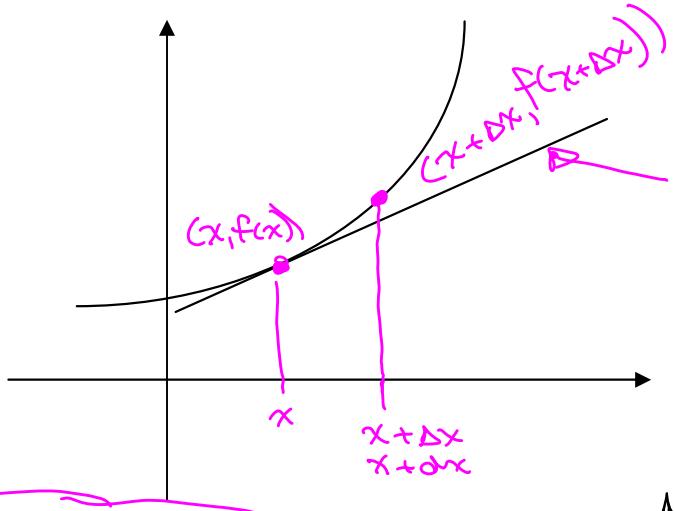
Example 2: Compute dy and Δy if $y = 5x + x^3$ as x changes from 3 to 3.05.

$$\begin{aligned} \frac{dy}{dx} &= 5+3x^2 \Rightarrow dy = (5+3x^2)dx \\ dy &\quad \left| \begin{array}{l} x=3 \\ dx=0.05 \end{array} \right. = (5+3(3)^2)(0.05) \\ &= 32(0.05) = \boxed{1.6} \\ \Delta y &= f(3.05) - f(3) \\ &= [5(3.05) + (3.05)^3] - [5(3) + 3^3] \\ &= 43.622625 - 42 \\ &= \boxed{1.622625} \end{aligned}$$

$$\begin{aligned} x + dx &= 10 - 0.3 = 9.7 \\ \Delta y &= f(x + dx) - f(x) \\ &= f(9.7) - f(10) \\ &= \sqrt{6+9.7} - \sqrt{6+10} \\ &= \sqrt{15.7} - \sqrt{16} = \sqrt{15.7} - 4 \\ &\approx \boxed{-0.037677} \end{aligned}$$

Compute actual
change in y :

The linearization of a function: (using a tangent line to approximate the function near a particular x -value.)



$$\text{slope of tangent line} = f'(x)$$

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

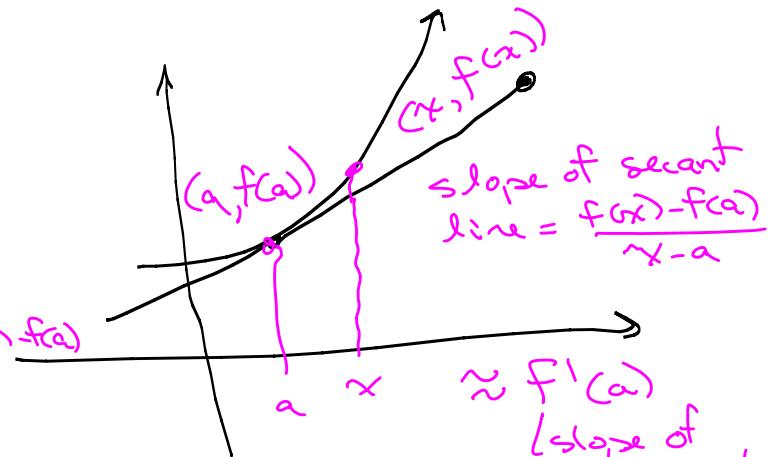
definition of derivative at $(a, f(a))$

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}$$

multiply both sides by $x - a$:

$$f(x) - f(a) \approx f'(a)(x - a)$$

$$f(x) \approx f(a) + f'(a)(x - a)$$



$$\text{slope of secant line} = \frac{f(x) - f(a)}{x - a}$$

$\approx f'(a)$
(slope of tangent line)

This tangent line gives us an approximation for the value of the function near a .

The linearization of f at a is

$$L(x) = f(a) + f'(a)(x - a)$$

$$y = y_1 + m(x - x_1)$$

The approximation $f(x) \approx L(x)$ is the standard linear approximation of f at a .

equation of tangent line.

Example 3: Find the linearization of $f(x) = x^4$ at 3. Use this to approximate $(3.013)^4$ and $(2.999)^4$.

$$\begin{aligned} y &= 108(3.013) - 243 \\ &= 82.404 \end{aligned}$$

$$\begin{aligned} f'(x) &= 4x^3 \\ \text{slope of tangent line: } m &= f'(3) = 4(3)^3 = 108 \\ \text{Find the } y\text{-value: } f(3) &= 3^4 = 81. \\ \text{Point: } (3, 81), \text{ slope } &= 108 \end{aligned}$$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 81 &= 108(x - 3) \\ y &= 108x - 324 + 81 \Rightarrow y = 108x - 243 \end{aligned}$$

Example 4: Find the linearization of $f(x) = \sin x$ at 60° . Use this to approximate $\sin 62^\circ$ and $\sin 58^\circ$.

$$f'(x) = \cos x$$

$$\text{At } x = 60^\circ, \text{ slope} = f'(60^\circ) = \cos 60^\circ = \frac{1}{2}$$

$$\text{Find point on tangent line: } f(60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\sqrt{3}}{2} = \frac{1}{2}(x - 60^\circ).$$

$$\text{Linearization: } L(x) = \frac{1}{2}(x - 60^\circ) + \frac{\sqrt{3}}{2}, \text{ where } x \text{ is measured in degrees.}$$

$$L(62^\circ) = \frac{1}{2}(62^\circ - 60^\circ) + \frac{\sqrt{3}}{2}$$

$$\begin{aligned} &= \frac{2^\circ}{2} + \frac{\sqrt{3}}{2} \\ &= \frac{2^\circ}{2} \left(\frac{\pi}{180^\circ}\right) + \frac{\sqrt{3}}{2} \quad (\text{Need to change it to a real #, otherwise it does not make sense to add it to } \frac{\sqrt{3}}{2}) \\ &= \frac{\pi}{180} + \frac{\sqrt{3}}{2} \approx 0.88348 \end{aligned}$$

Compare to calculating $\sin 62^\circ$ on calculator: $\sin 62^\circ \approx 0.8829476$

$$L(58^\circ) = \frac{1}{2}(58^\circ - 60^\circ) + \frac{\sqrt{3}}{2} = -\frac{2^\circ}{2} \left(\frac{\pi}{180^\circ}\right) + \frac{\sqrt{3}}{2} = -\frac{\pi}{180} + \frac{\sqrt{3}}{2} \approx 0.848572$$

Error propagation: Calculating $\sin 58^\circ$ on calculator results in 0.84804809

If x is the measured value of a variable and $x + \Delta x$ is the exact value of the variable, then Δx is the measurement error. If we use the measured value of x to calculate the value of a function f , then the propagated error is $\Delta y = f(x + \Delta x) - f(x)$. The propagated error can be estimated by calculating $dy = f'(x)dx \approx f'(x)\Delta x$.

Estimating propagated error:

If x is the measured value of a variable and Δx is the measurement error, then:

Estimated propagated error: $dy = f'(x)dx \approx f'(x)\Delta x$

Estimated relative error: $\frac{dy}{y}$

Example 5: The measurement of the radius of a circle is 20 inches with a maximum error of 0.10 inch. Approximate the maximum propagated error and the relative error in computing the area and the circumference of the circle.

Area

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

Propagated error = dA

Circumference

$$C = 2\pi r$$

$$\frac{dC}{dr} = 2\pi$$

$$dC = 2\pi dr$$

Prop error: $dC = 2\pi(0.10)$

$$\begin{aligned} r &= 20 \\ dr &= 0.10 \end{aligned}$$

$$= 0.2\pi \approx 0.628 \text{ in}$$

Relative error: $\frac{dC}{C} = \frac{0.2\pi}{2\pi r} = \frac{1}{10r}$

$$= 2\pi(20)(0.10) = 4\pi \text{ in}^2 \approx 12.57 \text{ in}^2$$

Relative error: $\frac{dA}{A} = \frac{12.57 \text{ in}^2}{\pi(20 \text{ in})^2} \approx 0.01 \Rightarrow 1\% \text{ error}$

$$= \frac{1}{10(20)} = \frac{1}{200} = 0.5\% \Rightarrow 0.5\%$$

Example 6: The measurements of the height and inside radius of a right cylinder are 50 feet and 30 feet, respectively. The maximum possible error in each measurement is about 3 inches per each 10 feet of measured length. Approximate the maximum propagated error and the relative error in computing the volume of the cylinder.

$V = \pi r^2 h$ We'll need the product rule, because we have both h and r .

$$\frac{dV}{dr} = \pi r^2 \frac{dh}{dr} + h(2\pi r)$$

Multiply both sides by dr :

$$dV = \pi r^2 dh + 2\pi r h dr$$

For h , measurement error dh is $5(3) = 15 \text{ in} = 15 \text{ in} \left(\frac{1 \text{ ft}}{12 \text{ in}}\right) = 1.25 \text{ ft}$

For r , measurement error dr is $3(3) = 9 \text{ in} = 9 \text{ in} \left(\frac{1 \text{ ft}}{12 \text{ in}}\right) = 0.75 \text{ ft}$

$$\begin{aligned} dV &= \pi(30 \text{ ft})^2(1.25 \text{ ft}) + 2\pi(30 \text{ ft})(50 \text{ ft})(0.75 \text{ ft}) \\ &= 11250 \text{ ft}^3 + 22500 \text{ ft}^3 = 3375\pi \text{ ft}^3 \end{aligned}$$

$$V = \pi r^2 h \Rightarrow V = \pi(30 \text{ ft})^2(50 \text{ ft}) = 45000\pi \text{ ft}^3$$

$$\text{Relative error} = \frac{dV}{V} = \frac{3375\pi \text{ ft}^3}{45000\pi \text{ ft}^3} = \frac{3375}{45000} = 0.075 \Rightarrow 7.5\%$$