4.1: Antiderivatives and Indefinite Integration

Definition: An *antiderivative* of *f* is a function whose derivative is *f*.

i.e. A function *F* is an antiderivative of *f* if $F'(x) = f(x)$.

Example 1: $x^3 + 5x$ is an antiderivative of $f(x) = 3x^2 + 5$.

What are some more antiderivatives of $f(x) = 3x^2 + 5$?

$x^3 + 5x + 1$

$x^3 + 5x - 1$

$x^3 + 5x + C$

So we have a whole "family" of antiderivatives of *f*.

Definition: A function *F* is called an antiderivative of *f* on an interval *I* if $F'(x) = f(x)$ for all *x* in *I*.

Theorem: If *F* is an antiderivative of *f* on an interval *I*, then all antiderivatives of *f* on *I* will be of the form

$$F(x) + C$$

where *C* is an arbitrary constant.

Example 2: Find the general form of the antiderivatives of $f(x) = 3x^2 + 5$.

$F(x) = x^3 + 5x + C$

Example 3: Find the general form of the antiderivatives of $f(x) = 6x^5 + \cos x$.

$F(x) = x^6 + \sin x + C$

Integration:

<u>Integration</u> is the process of finding antiderivatives.

> $\int f(x)dx$ is called the *indefinite integral* of *f*.
>
> $\int f(x)dx$ is the family of antiderivatives, or the most general antiderivative of *f*.
>
> This means: $\int f(x)dx = F(x) + c$, where $F'(x) = f(x)$.
>
> The *c* is called the *constant of integration*.

Integral sign

"with respect to *x*"

$$\int f(x)dx$$

Integrand

<u>Example 4:</u> Find $\int 3x^2 + 5\, dx$.

$$\boxed{x^3 + 5x + C}$$

<u>Example 5:</u> Find $\int 6x^5 + \cos x\, dx$.

$$\boxed{x^6 + \sin x + C}$$

<u>Example 6:</u> Find $\int \sec^2 dx$.

$$\boxed{\tan x + C}$$

$\int \sec^2 x dx$

Rules for Finding Antiderivatives:

Notation in this table: F is an antiderivative of f, G is an antiderivative of g,

Function	Antiderivative
k	$kx+c$
$kf(x)$	$kF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
x^n for $n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

add 1 to exp
divide by new

$\int x^4 dx = \frac{x^5}{5} + c$

1. $\int k \; dx = kx + c \quad$ (k a constant)

2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$

3. $\int k \; f(x) \; dx = k \int f(x) \; dx \quad$ (k a constant)

4. $\int [f(x)+g(x)] dx = \int f(x) dx + \int g(x) dx$

5. $\int \cos x \, dx = \sin x + c$

6. $\int \sin x \, dx = -\cos x + c$

7. $\int \sec^2 x \, dx = \tan x + c$

8. $\int \sec x \tan x \, dx = \sec x + c$

Example 7: Find the general antiderivative of $f(x) = \frac{1}{2}$.

$F(x) = \frac{1}{2}x + c$

Example 8: Find $\int x^3\,dx$.

$$\frac{x^4}{4} + C$$

Example 9: Find $\int 7x^2\,dx$.

$$\frac{7x^3}{3} + C$$

$$\int 3x^2\,dx$$

$$\frac{3x^3}{3} + C$$

$$x^3 + C$$

Example 10: Find $\int \frac{1}{x^5}\,dx$.

$$\int x^{-5}\,dx = \frac{x^{-4}}{-4} + C = \frac{-1}{4x^4} + C$$

Example 11: Find the general antiderivative of $f(x) = \frac{5}{x^2}$.

$$\int 5x^{-2}\,dx = \frac{5x^{-1}}{-1} + C = \frac{-5}{x} + C$$

Example 12: $\int \left(6x^2 - 3x + 9\right) dx$

$$= \frac{6x^3}{3} - \frac{3x^2}{2} + 9x + C = 2x^3 - \frac{3}{2}x^2 + 9x + C$$

Example 13: Find $\int 3\sqrt{x}\,dx$.

$$\int 3x^{1/2}\,dx = \frac{2}{3} \cdot 3x^{3/2} + C = 2x^{3/2} + C$$

$$\frac{1}{3} - 5 = \frac{1}{3} - \frac{15}{3} = -14/3$$

Example 14: Find $\int (3\cos x + 5\sin x)\, dx$.

$$3\sin x - 5\cos x + C$$

Example 15: Find $\int \frac{x^7 - \sqrt[3]{x} + 3x^2}{x^5}\, dx$.

$$= \int \frac{x^7}{x^5} - \frac{x^{1/3}}{x^5} + \frac{3x^2}{x^5}\, dx$$

$$= \int x^2 - x^{-14/3} + 3x^{-3} = \frac{x^3}{3} + \frac{3}{11}x^{-11/3} - \frac{3x^{-2}}{2} + C$$

$$x^3/3 + 3/11x^{11/3} - 3/2x^2 + C$$

Example 16: Find the general antiderivative of $f(\theta) = \frac{\sin\theta}{3}$.

$$-\frac{1}{3}\cos\theta + C$$

Example 17: $\int \left(\sqrt[3]{x} + \frac{2}{\sqrt{x}}\right) dx = \int x^{1/3} + 2x^{-1/2}\, dx$

$$\frac{3}{4}x^{4/3} + 2 \cdot 2x^{1/2} + C = \frac{3}{4}x^{4/3} + 4x^{1/2} + C$$

Example 18: $\int (6y^2 - 2)(8y + 5)\, dy$

$$\int (48y^3 + 30y^2 - 16y - 10)\, dy$$

$$\frac{48}{4}y^4 + \frac{30}{3}y^3 - \frac{16}{2}y^2 - 10y + C$$

$$12y^4 + 10y^3 - 8y^2 - 10y + C$$

Differential equations:

A *differential equation* is an equation involving the derivative of a function. To solve a differential equation means to find the original function.

An *initial value problem* is a common type of differential equation in which a derivative and an initial condition are given.

Example 19: Given $f'(x) = x^2 - 7$, find *f*. This is an example of a differential equation.

$$\int x^2 - 7\,dx = \frac{x^3}{3} - 7x + C$$

Example 20: Suppose that $f'(x) = 3x^2 + 2\cos x$ and $f(0) = 3$. Find $f(x)$.

$$\int (3x^2 + 2\cos x)\,dx = x^3 + 2\sin x + C$$

$$f(x) = x^3 + 2\sin x + C \qquad \boxed{x^3 + 2\sin x + 3}$$

$$3 = 0^3 + 2\sin(0) + C$$

$$3 = 0 + 0 + C \qquad C = 3$$

Example 21: Suppose that $f''(x) = 2x^3 - 6x^2 + 6x$, $f'(2) = -1$, and $f(-1) = 4$. Find $f(x)$.

$\int (2x^3 - 6x^2 + 6x)\, dx$

$f'(x) = \frac{1}{2}x^4 - 2x^3 + 3x^2 + C$

$-1 = \frac{1}{2}(2)^4 - 2(2)^3 + 3(2)^2 + C$

$-1 = 8 - 16 + 12 + C$

$-1 = 4 + C$

$-5 = C$

$\int \frac{1}{2}x^4 - 2x^3 + 3x^2 - 5\, dx$

$\frac{1}{10}x^5 - \frac{1}{2}x^4 + x^3 - 5x + D = f(x)$

$4 = -\frac{1}{10} - \frac{1}{2} - 1 + 5 + D$

$\frac{40}{10} = -\frac{1}{10} - \frac{5}{10} + \frac{40}{10} + D$

$\frac{3}{5} = \frac{6}{10} = D$

$f(x) = \frac{1}{10}x^5 - \frac{1}{2}x^4 + x^3 - 5x + 3/5$

Example 22: Suppose that $f''(x) = 12x^2 - 18x$, $f(1) = 2$, and $f(-3) = 1$. Find $f(x)$.

$f'(1) = 2$

$\int 12x^2 - 18x\, dx$

$4x^3 - 9x^2 + C = f'(x)$

$4(1)^3 - 9(1)^2 + C = 2$

$4 - 9 + C = 2$

$-5 + C = 2$

$C = 7$

$\int 4x^3 - 9x^2 + 7\, dx$

$f(x) = x^4 - 3x^3 + 7x + D$

$1 = (-3)^4 - 3(-3)^3 + 7(-3) + D$

$1 = 81 + 81 - 21 + D$

$1 = 141 + D$

$-140 = D$

$f(x) = x^4 - 3x^3 + 7x - 140$

Velocity and acceleration (rectilinear motion):

We already know that if $f(t)$ is the position of an object at time t, then $f'(t)$ is its velocity and $f''(t)$ is its acceleration.

Note: Acceleration due to gravity near the earth's surface is approximately 9.8 m/s^2 or 32 ft/s^2.

Example 23: Suppose a particle's velocity is given by $v(t) = 2\sin t + \cos t$ and its initial position is $s(0) = 3$. Find the position function of the particle.

$\int v(t)\,dt = s(t) + C$

$\int (2\sin t + \cos t)\,dt$

$-2\cos t + \sin t + C$

$3 = -2\cos(0) + \sin(0) + C$

$3 = -2(1) + C$

$5 = C$

$\boxed{-2\cos t + \sin t + 5}$

Example 24: Suppose a ball is thrown upward from a 30-foot bridge over a river at an initial velocity of 40 feet per second. How high does it go? When does it hit the water?

$a(t) = -32$

$\int -32\,dt$

$v(t) = -32t + C$

$40 = -32(0) + C$

$40 = C$

$v(t) = -32t + 40$

$\int -32t + 40\,dt$

$-16t^2 + 40t + D$

$30 = -16(0)^2 + 40(0) + D$

$30 = D$

$s(t) = -16t^2 + 40t + 30$

$\boxed{\text{55 feet max}}$

$0 = -16t^2 + 40t + 30$

$0 = -8t^2 + 20t + 15$

$8t^2 - 20t - 15 = 0$

$0 = -32t + 40$

$32t = 40$

$t = 40/32 = 1.25$

$s(1.25) = -16(1.25)^2 + 40(1.25) + 30 = 55$

$8t^2 - 20t - 15 = 0$

$$x = \frac{-(20) \pm \sqrt{400 - 4(8)(-15)}}{2(8)}$$

$$\frac{-20 \pm \sqrt{880}}{16} \qquad \frac{-20 \pm 29.665}{16}$$

$$= \frac{9.665}{16}$$

$\approx .6 sec$