4.4: The Fundamental Theorem of Calculus

Evaluating the area under a curve by calculating the areas of rectangles, adding them up, and letting taking the limit as $n \to \infty$ is okay in theory but is tedious at best and not very practical.

Fortunately, there is a theorem that makes calculating the area under the curve (definite integral) much easier.

> The Fundamental Theorem of Calculus:
>
> Let *f* be continuous on the interval $[a,b]$. Then,
>
> $$\int_a^b f(x)dx = F(b) - F(a)$$
>
> where *F* is any antiderivative of *f*; in other words, where $F'(x) = f(x)$.

Notation: We'll use this notation when evaluating definite integrals.

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example 1: Find the area under the graph of $f(x) = x$ between 0 and 3.

Notice that the constant C disappeared when we evaluated the definite integral. This will always happen.

$$\int_a^b f(x)dx = F(x)+c\Big|_a^b = \left(F(b)+c\right)-\left(F(a)+c\right) = F(b)+c-F(a)-c = F(b)-F(a)$$

So from now on, we'll omit the "$+c$" when evaluating definite integrals.

Example 2: Find the area under the graph of $f(x)=4x^2+1$ over the interval $[0,2]$. (Compare with our approximation in Section 4.2, Example 5).

Example 3: Evaluate $\int_{-2}^{4}(3x^2-x+4)dx$.

Example 4: Evaluate $\int_0^{\pi} (4x^3 + \cos x)dx$.

Example 5: Evaluate $\int_1^3 \left(\frac{3}{t^2} \right) dt$.

Example 6: Evaluate $\int_2^9 \frac{1}{\sqrt{u}} du$.

Example 7: Evaluate $\int_{-2}^4 \frac{1}{x^3} dx$

For now, if f has an infinite discontinuity anywhere in $[a,b]$, assume that $\int_a^b f(x)\,dx$ does not exist. Some of these integrals do exist….you will learn how to handle such integrals in Calculus 2.

<u>The Fundamental Theorem of Calculus, Part II</u>:

Let f be continuous on the interval $[a,b]$. Then the function g defined by

$$g(x) = \int_a^x f(t)\,dt\,, \qquad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

In other words, $\dfrac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$.

<u>Example 1:</u> Find the derivative of the function $f(x) = \int_3^x \dfrac{t^2 - 2t + 4}{t - 2}\,dt$.

<u>Example 2:</u> Find $\dfrac{d}{dx}\int_{-2}^{\sin x} \sqrt{t^4 + 2}\,dt$.

The mean (average) value of a function:

On the interval $[a,b]$, a continuous function $f(x)$ will have an average "height" *c* such that the rectangle with width $b-a$ and height *c* will have the same area as the area under the curve over $[a,b]$. This *c* is the *average value of the function f over* $[a,b]$.

> Mean Value Theorem for Integrals:
>
> If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that
>
> $$\int_a^b f(x)\,dx = f(c)(b-a)\,.$$

This number *c* is called the *average value* of the function *f* on the interval $[a,b]$.

> The *average value* of a continuous function *f* on the interval $[a,b]$ is given by
>
> $$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx\,.$$

Example 8: Find the average value of the function $f(x) = 4x^3 - x^2$ over the interval $[-3,2]$.

Example 9: Determine the average value of $f(x) = \sin x$ on the interval $[0,\pi]$.