

## 4.4: The Fundamental Theorem of Calculus

Evaluating the area under a curve by calculating the areas of rectangles, adding them up, and letting taking the limit as  $n \rightarrow \infty$  is okay in theory but is tedious at best and not very practical.

Fortunately, there is a theorem that makes calculating the area under the curve (definite integral) much easier.

The Fundamental Theorem of Calculus:

Let  $f$  be continuous on the interval  $[a, b]$ . Then,

$$\int_a^b f(x) dx = F(b) - F(a)$$

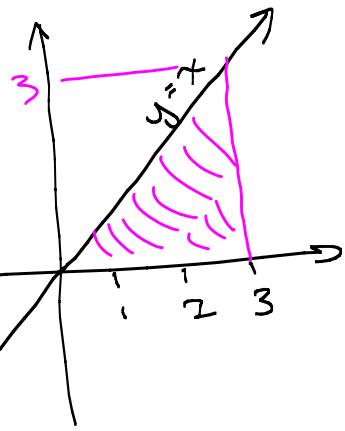
where  $F$  is any antiderivative of  $f$ ; in other words, where  $F'(x) = f(x)$ .

Notation: We'll use this notation when evaluating definite integrals.

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

Example 1: Find the area under the graph of  $f(x) = x$  between 0 and 3.

From Geometry:



$$\begin{aligned} \int_0^3 x dx &= \frac{1}{2}(3)(3) \\ &\quad \text{(base)(height)} \\ &= \boxed{\frac{9}{2}} \end{aligned}$$

Using the Fun. Theorem of Calculus

Note:  $\int x dx = \frac{x^2}{2} + C$   
(family of antiderivatives,  
or general antiderivatives)

$$\begin{aligned} \int_0^3 x dx &= \left( \frac{x^2}{2} + C \right) \Big|_{x=0}^{x=3} \\ &= \left( \frac{3^2}{2} + C \right) - \left( \frac{0^2}{2} + C \right) \\ &= \frac{9}{2} + C - 0 - C = \boxed{\frac{9}{2}} \end{aligned}$$

## 4.4.2

Notice that the constant  $C$  disappeared when we evaluated the definite integral. This will always happen.

$$\int_a^b f(x)dx = F(x) + C \Big|_a^b = (F(b) + C) - (F(a) + C) = F(b) - F(a)$$

So from now on, we'll omit the "+c" when evaluating definite integrals.

**Example 2:** Find the area under the graph of  $f(x) = 4x^2 + 1$  over the interval  $[0, 2]$ . (Compare with our approximation in Section 4.2, Example 5).

$$\int_0^2 (4x^2 + 1) dx = \left( \frac{4x^3}{3} + x \right) \Big|_0^2 = \left( \frac{4(2)^3}{3} + 2 \right) - \left( \frac{4(0)^3}{3} + 0 \right)$$

From Ex 5 in 4.2:

Right Endpts: 17  
Left Endpts: 9  
Midpts: 12.5

} approximations  
of the  
area

$$= \frac{32}{3} + 2 - 0 - 0 \\ = \boxed{\frac{38}{3}} = \boxed{12\frac{2}{3}}$$

Exact area

**Example 3:** Evaluate  $\int_{-2}^4 (3x^2 - x + 4)dx$ .

$$\begin{aligned} \int_{-2}^4 (3x^2 - x + 4)dx &= \left( \frac{3x^3}{3} - \frac{x^2}{2} + 4x \right) \Big|_{-2}^4 = \left( x^3 - \frac{x^2}{2} + 4x \right) \Big|_{-2}^4 \\ &= \left[ 4^3 - \frac{4^2}{2} + 4(4) \right] - \left[ (-2)^3 - \frac{(-2)^2}{2} + 4(-2) \right] = [64 - 8 + 16] - [-8 - 2 - 8] \\ &= 72 - [-18] = 72 + 18 = \boxed{90} \end{aligned}$$

Note:  $\int (3x^2 - x + 4)dx = x^3 - \frac{x^2}{2} + 4x + C$

Check:  $\frac{d}{dx} \left( x^3 - \frac{x^2}{2} + 4x \right) = 3x^2 - \frac{1}{2}(2x) + 4 = 3x^2 - x + 4$  ✓ok

Example 4: Evaluate  $\int_0^\pi (4x^3 + \cos x) dx$ .

$$\begin{aligned} \int_0^\pi (4x^3 + \cos x) dx &= \left( \frac{4x^4}{4} + \sin x \right) \Big|_0^\pi = (x^4 + \sin x) \Big|_0^\pi \\ &= [\pi^4 + \sin \pi] - [0^4 + \sin 0] = \pi^4 - 0 - 0 - 0 \\ &= \boxed{\pi^4} \end{aligned}$$

Example 5: Evaluate  $\int_1^3 \left( \frac{3}{t^2} \right) dt$ .

$$\begin{aligned} \int_1^3 3t^{-2} dt &= \frac{3t^{-1}}{-1} \Big|_1^3 = -\frac{3}{t} \Big|_1^3 = -\frac{3}{3} - \left( -\frac{3}{1} \right) \\ &= -1 + 3 = \boxed{2} \end{aligned}$$

Example 6: Evaluate  $\int_2^9 \frac{1}{\sqrt{u}} du$ .

$$\int_2^9 \frac{1}{\sqrt{u}} du = 2\sqrt{u} \Big|_2^9 = 2\sqrt{9} - 2\sqrt{2} = 2(3) - 2\sqrt{2} = \boxed{6 - 2\sqrt{2}}$$

$$\begin{aligned} \int \frac{1}{\sqrt{u}} du &= \int u^{-1/2} du = \frac{u^{-1/2+1}}{-1/2+1} + C \\ &= \frac{u^{1/2}}{1/2} + C = 2u^{1/2} + C = 2\sqrt{u} + C \\ \text{Check: } \frac{d}{du} (2\sqrt{u}) &= 2(\frac{1}{2})u^{-1/2} = \frac{1}{\sqrt{u}} \end{aligned}$$

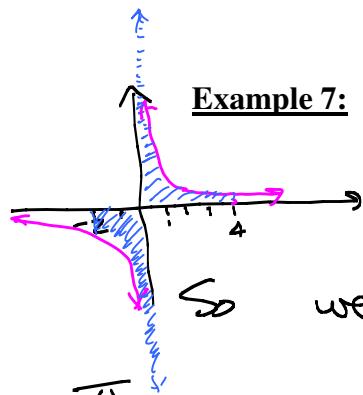
Example 7: Evaluate  $\int_{-2}^4 \frac{1}{x^3} dx$

$\frac{1}{x^3} \rightarrow$  not continuous on

$[[-2, 4]]$ . Discontinuous at  $x=0$ .

So we cannot apply the Fun. Thm. of Calculus

This is an example of an improper integral. Some improper integrals can be evaluated... we'll do this in Calculus II.



For now, if  $f$  has an infinite discontinuity anywhere in  $[a, b]$ , assume that  $\int_a^b f(x) dx$  does not exist. Some of these integrals do exist....you will learn how to handle such integrals in Calculus 2.

### The Fundamental Theorem of Calculus, Part II:

Let  $f$  be continuous on the interval  $[a, b]$ . Then the function  $g$  defined by

$$g(x) = \int_a^x f(t) dt, \quad a \leq x \leq b$$

*g is the area under the curve f from a to x*

is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and  $\underline{g'(x)} = f(x)$ .

$$\text{In other words, } \frac{d}{dx} \left[ \int_a^x f(t) dt \right] = f(x).$$

**Example 1:** Find the derivative of the function  $\underline{g(x)} = \int_3^x \frac{t^2 - 2t + 4}{t-2} dt$ .

*(this f corresponds to g in the theorem)*

$f(t) = \frac{t^2 - 2t + 4}{t-2}$  is continuous on  $(3, \infty)$

$$g'(x) = \frac{d}{dx} \left[ \int_3^x \frac{t^2 - 2t + 4}{t-2} dt \right] = \boxed{\frac{x^2 - 2x + 4}{x-2}}$$

**Example 2:** Find  $\frac{d}{dx} \left( \int_{-2}^{\sin x} \sqrt{t^4 + 2} dt \right)$ .  $f(t) = \sqrt{t^4 + 2}$  is continuous on  $(-\infty, \infty)$

Area =  $A = \int_{-2}^{\sin x} \sqrt{t^4 + 2} dt$ . I want to find  $\frac{dA}{dx}$ .

Let  $u = \sin x$ .

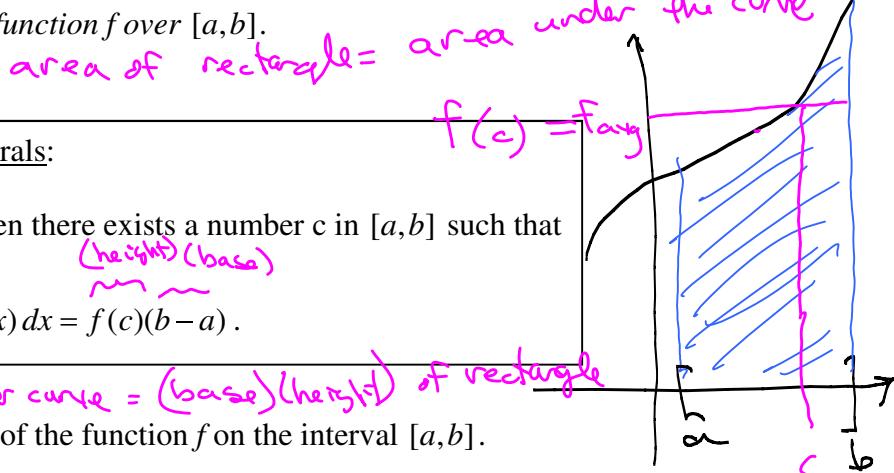
$$\text{Then } A = \int_{-2}^u \sqrt{t^4 + 2} dt$$

$$\frac{dA}{du} = f(u) = \sqrt{u^4 + 2}$$

$$\begin{aligned} \text{Chain Rule: } \frac{dA}{dx} &= \frac{dA}{du} \cdot \frac{du}{dx} = (\sqrt{u^4 + 2})(\cos x) = (\sqrt{(\sin x)^4 + 2})(\cos x) \\ &= \boxed{\cos x \sqrt{\sin^4 x + 2}} \end{aligned}$$

### The mean (average) value of a function:

On the interval  $[a, b]$ , a continuous function  $f(x)$  will have an average "height"  $c$  such that the rectangle with width  $b - a$  and height  $c$  will have the same area as the area under the curve over  $[a, b]$ . This  $c$  is the *average value of the function  $f$  over  $[a, b]$* .



#### Mean Value Theorem for Integrals:

If  $f$  is continuous on  $[a, b]$ , then there exists a number  $c$  in  $[a, b]$  such that

$$\int_a^b f(x) dx = f(c)(b-a).$$

The  $y$ -value  $f(c)$  is Area under curve = (base)(height) of rectangle  
This number  $c$  is called the *average value* of the function  $f$  on the interval  $[a, b]$ .

$$\begin{aligned} \int_a^b f(x) dx &= f(c)(b-a) \\ \frac{1}{b-a} \int_a^b f(x) dx &= f(c) = f_{avg} \end{aligned}$$

The *average value* of a continuous function  $f$  on the interval  $[a, b]$  is given by

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx.$$

**Example 8:** Find the average value of the function  $f(x) = 4x^3 - x^2$  over the interval  $[-3, 2]$ .

$$\begin{aligned} f_{avg} &= \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{1}{2 - (-3)} \int_{-3}^2 (4x^3 - x^2) dx \\ &= \frac{1}{5} \left[ \frac{4x^4}{4} - \frac{x^3}{3} \right]_{-3}^2 \\ &\stackrel{\rightarrow}{=} \frac{1}{5} \left[ x^4 - \frac{x^3}{3} \right]_{-3}^2 \\ &= \frac{1}{5} \left[ 2^4 - \frac{2^3}{3} - (-3)^4 - \frac{(-3)^3}{3} \right] \\ &= \frac{1}{5} \left[ 16 - \frac{8}{3} - 81 + 9 \right] \\ &= \frac{1}{5} \left[ -74 - \frac{8}{3} \right] \\ &= \frac{1}{5} \left[ -\frac{222}{3} - \frac{8}{3} \right] \\ &= \frac{1}{5} \left( -\frac{230}{3} \right) \\ &= -\frac{230}{15} \\ &= -\frac{46}{3} \end{aligned}$$

**Example 9:** Determine the average value of  $f(x) = \sin x$  on the interval  $[0, \pi]$ .

$$\begin{aligned} f_{avg} &= \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{1}{\pi - 0} \int_0^\pi \sin x dx \\ &= \frac{1}{\pi} \left[ -\cos x \right]_0^\pi = \frac{1}{\pi} \left[ -\cos \pi - (-\cos 0) \right] \\ &= \frac{1}{\pi} \left[ -\cos \pi + \cos 0 \right] = \frac{1}{\pi} [-(-1) + 1] \\ &= \frac{1}{\pi} [1 + 1] = \boxed{\frac{2}{\pi}} \end{aligned}$$