

## 4.5: Integration by Substitution

Hugely important section,  
esp. for Calc. 4.5.1

Most functions cannot be integrated using only the formulas we have learned so far. In Calculus II, you will learn several advanced integration techniques. For now, we'll learn just one new integration technique, called substitution.

**Example 1:** Find  $\int 4(4x-9)^7 dx$ .

One way to do this would be to multiply it out into a long polynomial....YUK!

Here's another way:

$$\int 4(4x-9)^7 dx = \int u^7 du \quad \left| \begin{array}{l} \text{check} \\ \frac{d}{dx} \left( \frac{1}{8}(4x-9)^8 + C \right) \\ = \frac{1}{8}(8)(4x-9)^7(4) + D \\ = 4(4x-9)^7 \text{ over } \\ \frac{(4x-9)^8}{8} + C \end{array} \right. \quad \left\{ \begin{array}{l} \text{Let} \\ u = 4x-9 \\ \frac{du}{dx} = 4 \\ du = 4 dx \end{array} \right.$$

### The Substitution Rule:

If  $u = g(x)$  is a differentiable function whose range is an interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

**Example 2:** Find  $\int 6x^2(2x^3+5)^4 dx$ .

$$\int 6x^2(2x^3+5)^4 dx = \int u^4 du = \frac{u^5}{5} + C \quad \left| \begin{array}{l} \text{check} \\ \frac{d}{dx} \left( \frac{(2x^3+5)^5}{5} + C \right) \\ = (2x^3+5)^4 \cdot 6x^2 \end{array} \right. \quad \left\{ \begin{array}{l} u = 2x^3+5 \\ \frac{du}{dx} = 6x^2 \\ du = 6x^2 dx \end{array} \right.$$

**Example 3:** Find  $\int -2\sqrt{-2x-4} dx$ .

$$\int -2(-2x-4)^{1/2} dx = \int u^{1/2} du = \frac{u^{3/2}}{3/2} + C \quad \left| \begin{array}{l} \text{check} \\ \frac{d}{dx} \left( \frac{2}{3}(-2x-4)^{3/2} + C \right) \\ = -2(-2x-4)^{1/2} \end{array} \right. \quad \left\{ \begin{array}{l} u = -2x-4 \\ \frac{du}{dx} = -2 \\ du = -2 dx \end{array} \right.$$

or  $\int x^3(x^4-1)^6 dx = \frac{1}{4} \int 4x^3(x^4-1)^6 dx = \frac{1}{4} \int u^6 du = \frac{1}{4} \cdot \frac{u^7}{7} + C$

$$= \frac{u^7}{28} + C = \boxed{\frac{(x^4-1)^7}{28} + C}$$

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Example 4: Find  $\int x^3(x^4-1)^6 dx$ .

$$\int x^3(x^4-1)^6 dx = \int \frac{1}{4} u^6 du = \frac{1}{4} \int u^6 du$$

$$= \frac{1}{4} \cdot \frac{u^7}{7} + C = \frac{u^7}{28} + C$$

$$= \boxed{\frac{(x^4-1)^7}{28} + C}$$

Check:  $\frac{d}{dx} \left( \frac{1}{28} (x^4-1)^7 \right) = \frac{1}{28} (-)(x^4-1)^6 (4x^3) = x^3(x^4-1)^6 \checkmark$

Example 5: Find  $\int \sin 5t dt$ .

$$\int \underbrace{\sin(5t)}_{\sin u} \frac{1}{5} du = \int \frac{1}{5} \sin u du = \frac{1}{5} \int \sin u du$$

$$= \frac{1}{5} (-\cos u) + C$$

$$= \boxed{-\frac{1}{5} \cos 5t + C}$$

Try  $\sin 5t = u$   
 $\frac{du}{dt} = 5 \cos 5t$   
 $du = 5 \cos 5t dt$  doesn't show up in the integrand

Example 6: Find  $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$ .

$$\left. \begin{aligned} u &= x^4 - 1 \\ \frac{du}{dx} &= 4x^3 \\ du &= 4x^3 dx \\ \frac{1}{4} du &= x^3 dx \end{aligned} \right\}$$

$$\left. \begin{aligned} u &= 5t \\ \frac{du}{dt} &= 5 \\ du &= 5dt \\ \frac{1}{5} du &= dt \end{aligned} \right\}$$

4.5.3

Example 7: Find  $\int \frac{x}{\sqrt[3]{5x^2 - 8}} dx$ .

$$\begin{aligned} \int x (5x^2 - 8)^{-1/3} dx &= \frac{1}{10} \int u^{-1/3} du \\ &= \frac{1}{10} \cdot \frac{u^{2/3}}{2/3} + C \\ &= \frac{1}{10} \cdot \frac{3}{2} u^{2/3} + C = \frac{3}{20} u^{2/3} + C \\ &= \boxed{\frac{3}{20} (5x^2 - 8)^{2/3} + C} \end{aligned}$$

$u = 5x^2 - 8$

$\frac{du}{dx} = 10x$

$du = 10x dx$

$\frac{1}{10} du = x dx$

Example 8: Find  $\int \cos x \sin^5 x dx$ 

$$\begin{aligned} \int (\cos x) (\sin x)^5 dx &= \int u^5 du = \frac{u^6}{6} + C \\ &= \boxed{\frac{1}{6} \sin^6 x + C} \end{aligned}$$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x dx$

Example 9: Find  $\int \sec^2 \left( \frac{x}{7} \right) dx$ .

$$\begin{aligned} \int \sec^2 \left( \frac{x}{7} \right) dx &= 7 \int \sec^2 u du = 7 \tan u + C \\ &= \boxed{7 \tan \left( \frac{x}{7} \right) + C} \end{aligned}$$

Note:  $\frac{d}{dx} (\tan x) = \sec^2 x \Rightarrow \int \sec^2 x dx = \tan x + C$

$u = \frac{x}{7} = \frac{1}{7} x$

$\frac{du}{dx} = \frac{1}{7}$

$du = \frac{1}{7} dx$

$7 du = dx$

Example 10: Find  $\int x(5+x)^4 dx$ .

$$\begin{aligned} \int x (5+x)^4 dx &= \int x u^4 du \\ &= \int (u-5)(u^4) du \\ &= \int (u^5 - 5u^4) du \\ &= \frac{u^6}{6} - \frac{5u^5}{5} + C = \frac{(5+x)^6}{6} - (5+x)^5 + C \\ &= (5+x)^5 \left( \frac{5+x-6}{6} \right) + C = (5+x)^5 \left( \frac{s+x-6}{6} \right) + C \end{aligned}$$

$u = 5+x$

$\frac{du}{dx} = 1$

$du = dx$

$u = 5+x \Rightarrow u-5=x$

$$\begin{aligned}
 & \text{check: } \frac{d}{dx} \left[ \frac{(s+x)^5}{5} - (s+x)^5 \right] = \frac{1}{5} (s+x)^4 (s+1) \\
 & \quad - s(s+x)^4 (1) = (s+x)^4 [s+x - 5] = (s+x)^4 (x) = x(s+x)^4 \text{ ✓OK} \\
 & = (s+x)^5 - 5(s+x)^4 = (s+x)^4 [s+x - 5] = (s+x)^4 (x) = x(s+x)^4 \text{ ✓OK}
 \end{aligned} \tag{4.5.4}$$

**Definite integrals:**

The Substitution Rule for Definite Integrals:

If  $g'$  is continuous on  $[a, b]$  and  $f$  is continuous on the range of  $u = g(x)$ , then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

There are two methods of evaluating definite integrals when substitution is involved:

Method 1: Find the antiderivative using substitution and switch back to your original variable. Then evaluate the definite integral using the original upper and lower limits.

Method 2: (Using the Substitution Rule for Definite Integrals) Find the antiderivative using substitution, but don't switch back to your original variable. Instead, calculate the upper and lower limits in terms of  $u$  (or whatever variable you used to substitute). Then evaluate your definite integral using these "new" upper and lower limits.

Example 11: Evaluate  $\int_{-1}^2 x^4(2x^5 - 8)^3 dx$  using both methods.

Method 1:

Find antiderivative:

$$\int x^4(2x^5 - 8)^3 dx$$

$$\begin{aligned}
 & u = 2x^5 - 8 \\
 & du = 10x^4 dx \\
 & \frac{1}{10} du = x^4 dx
 \end{aligned}$$

$$\begin{aligned}
 & = \frac{1}{10} \int u^3 du \\
 & = \frac{1}{10} \cdot \frac{u^4}{4} + C \\
 & = \frac{1}{40} (2x^5 - 8)^4 + C
 \end{aligned}$$

$$\begin{aligned}
 & u = 2x^5 - 8 \\
 & x=2 \Rightarrow u=2(2^5)-8 \\
 & = 64-8=56
 \end{aligned}$$

$$\begin{aligned}
 & x=-1 \Rightarrow u=2(-1)^5-8 \\
 & = -2-8 \\
 & = -10
 \end{aligned}$$

$$\begin{aligned}
 \int_{-1}^2 x^4(2x^5 - 8)^3 dx &= \frac{1}{40} (2x^5 - 8)^4 \Big|_{-1}^2 \\
 &= \frac{1}{40} (2(2^5 - 8)^4) - \frac{1}{40} (2(-1)^5 - 8)^4 \\
 &= \frac{1}{40} (64 - 8)^4 - \frac{1}{40} (-2 - 8)^4 \\
 &= \frac{56^4}{40} - \frac{(-10)^4}{40} = \frac{9834496}{40} - \frac{10000}{40} \\
 &= 245612.4
 \end{aligned}$$

Method 2: (changing the limits of integration)

$$\int_{-1}^2 x^4(2x^5 - 8)^3 dx$$

$$\begin{aligned}
 & u = 2x^5 - 8 \\
 & u = -10 \quad u = 56
 \end{aligned}$$

$$\begin{aligned}
 & = \frac{1}{10} \int u^3 du \Big|_{-10}^{56} \\
 & u = -10 \quad u = 56
 \end{aligned}$$

$$\begin{aligned}
 & = \frac{1}{10} \left( (56)^4 - (-10)^4 \right) \\
 & = \frac{1}{10} (9834496 - 10000) \\
 & = 245612.4
 \end{aligned}$$

Note:  $3-5x=0 \Rightarrow x=\frac{3}{5}$  only discontinuity is not in interval  $[-4, -2]$

I'll work this by changing the limits of integration. (Method 2) 4.5.5

Example 12: Evaluate  $\int_{-4}^{-2} \frac{4}{(3-5x)^3} dx$

$$\begin{aligned} \int_{-4}^{-2} \frac{4}{(3-5x)^3} dx &= 4 \int_{-4}^{-2} (3-5x)^{-3} dx \\ &= 4 \left(-\frac{1}{3}\right) \int_{23}^{13} u^{-3} du = -\frac{4}{5} \cdot \frac{u^{-2}}{-2} \Big|_{u=23}^{u=13} \\ &= -\frac{4}{10u^2} \Big|_{23}^{13} = \frac{2}{5u^2} \Big|_{23}^{13} = \\ &= \frac{2}{5(13)^2} - \frac{2}{5(23)^2} \approx 0.0016107 \\ &= \frac{144}{89401} \end{aligned}$$

$$\begin{cases} u = 3-5x \\ \frac{du}{dx} = -5 \\ du = -5 dx \\ -\frac{1}{5} du = dx \\ x = -2 \Rightarrow u = 3-5(-2) \\ = 3+10 \\ = 13 \\ x = -4 \Rightarrow u = 3-5(-4) \\ = 3+20 \\ = 23 \end{cases}$$

Example 13: Evaluate  $\int_{\pi/4}^{\pi/3} \frac{\sin x}{\cos^3 x} dx$ .

$$\begin{aligned} \int_{\pi/4}^{\pi/3} \frac{\sin x}{(\cos x)^3} dx &= - \int_{\pi/2}^{1/2} \frac{1}{u^3} du \\ &= - \int_{\pi/2}^{1/2} u^{-3} du \\ &= -\frac{u^{-2}}{-2} \Big|_{\pi/2}^{1/2} = \frac{1}{2u^2} \Big|_{\pi/2}^{1/2} \\ &= \frac{1}{2(1/2)^2} - \frac{1}{2(\pi/2)^2} = \frac{1}{2/1} - \frac{1}{2(\pi/2)} \\ &= \frac{4}{2} - \frac{1}{\pi} = 2 - 1 = 1 \end{aligned}$$

$$\begin{cases} u = \cos x \\ \frac{du}{dx} = -\sin x \\ du = -\sin x dx \\ -du = \sin x dx \\ x = \frac{\pi}{3} \Rightarrow u = \cos \frac{\pi}{3} = \frac{1}{2} \\ x = \frac{\pi}{4} \Rightarrow u = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \end{cases}$$

## Symmetry and definite integrals:

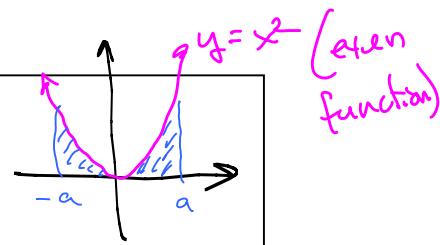
Suppose  $f$  is continuous on  $[-a, a]$ .

*y-axis*

(a) If  $f$  is even (symmetric about *x-axis*,  $f(-x) = f(x)$ ), then

$$\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx.$$

(b) If  $f$  is odd (symmetric about origin,  $f(-x) = -f(x)$ ), then  $\int_{-a}^a f(x)dx = 0$ .

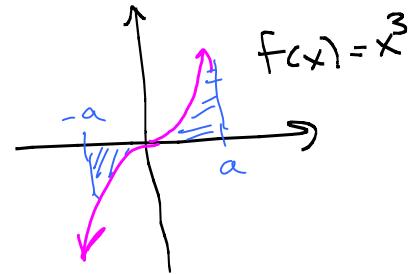


Example 14:  $\int_{-3}^3 x^2 dx = 2 \int_0^3 x^2 dx$

(because  $f(x) = x^2$  is an even function)

$$= 2 \frac{x^3}{3} \Big|_0^3$$

$$= 2 \left( \frac{3^3}{3} - \frac{0^3}{3} \right) = 2(9 - 0) = 18$$



Example 15: Find  $\int_{-13}^{13} 7x^5 dx$

Notice:  $f(x) = 7x^5$  is an odd function.

$$\int_{-13}^{13} 7x^5 dx = 0$$

Ex:  $\int (4x-3)^4 dx$

$u = 4x-3$   
 $\frac{du}{dx} = 4$   
 $du = 4 dx$   
 $\frac{1}{4} du = dx$

$= \frac{1}{4} \cdot \frac{u^5}{5} + C = \frac{u^5}{20} + C = \boxed{\frac{1}{20} (4x-3)^5 + C}$

Check:  $\frac{d}{dx} \left[ \frac{1}{20} (4x-3)^5 \right] = \frac{1}{20} (5)(4x-3)^4 (4)$   
 $= (4x-3)^4 \checkmark_{OK}$

Ex:  $\int \cos(8x) dx$

$u = 8x$   
 $\frac{du}{dx} = 8$   
 $du = 8 dx$   
 $\frac{1}{8} du = dx$

$= \frac{1}{8} \int \cos u du = \frac{1}{8} \sin u + C$   
 $= \boxed{\frac{1}{8} \sin 8x + C}$

Check:  $\frac{d}{dx} \left( \frac{1}{8} \sin 8x \right) = \frac{1}{8} \cdot (\cos 8x)(8) = \cos 8x \checkmark_{OK}$

Note: For every derivative we know, we also know the corresponding antiderivative

$$\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x \, dx = \tan x + C$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x \, dx = \sec x + C$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \Rightarrow \frac{d}{dx}(-\cot x) = \csc^2 x$$

$$\Rightarrow \int \csc^2 x \, dx = -\cot x + C$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x \Rightarrow \int \csc x \cot x \, dx = -\csc x + C$$

$$\frac{d}{dx}(\cos x) = -\sin x \Rightarrow \int \sin x \, dx = -\cos x + C$$

$$\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x \, dx = \sin x + C$$

## Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\text{Divide by } \cos^2 \theta: \quad \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\text{Divide by } \sin^2 \theta: \quad \frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

$$1 + \cot^2 \theta = \csc^2 \theta$$