5.1: The Natural Logarithmic Function: Differentiation

An algebraic approach to logarithms:

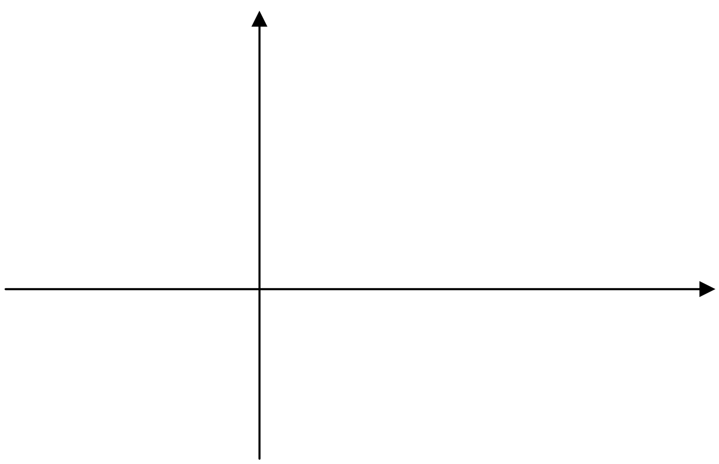
> Definition: $\log_b x = y$ is equivalent to $b^y = x$.
>
> The functions $f(x) = b^x$ and $g(x) = \log_b x$ are inverses of each other.
> b is called the *base* of the logarithm.

The logarithm of base e is called the *natural logarithm*, which is abbreviated "ln".

> The natural logarithm:
>
> $$\ln x = \log_e x.$$
>
> Therefore $\ln x = y$ is equivalent to $e^y = x$ and the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverses of each other.

A calculus approach to the natural logarithm:

> The natural logarithm function is defined as
>
> $$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$



For $x > 1$, $\ln x$ can be interpreted as the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to $t = x$.

Note: The integral is not defined for $x < 0$.

For $x = 1$, $\ln x = \int_1^1 \frac{1}{t}\,dt = 0$.

For $x < 1$, $\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$.

Recall:

The Fundamental Theorem of Calculus, Part II:

Let *f* be continuous on the interval $[a,b]$. Then the function g defined by

$$g(x) = \int_a^x f(t)\,dt\,, \qquad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

Apply the Fundamental Theorem of Calculus to the function $f(t) = \frac{1}{t}$.

$$\frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x}$$

This means that $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

The Derivative of the Natural Logarithmic Function

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Laws of Logarithms:

If *x* and *y* are positive numbers and *r* is a rational number, then:

1. $\ln(xy) = \ln x + \ln y$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

   Note: This also gives us $\ln\left(\frac{1}{x}\right) = -\ln x$.

3. $\ln\left(x^r\right) = r \ln x$

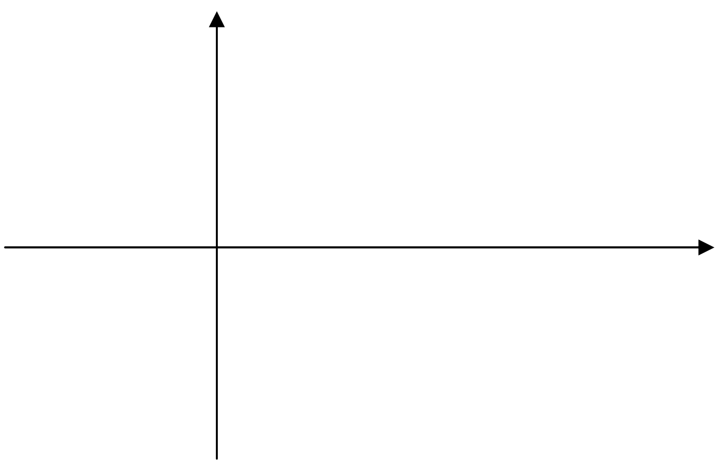
Example 1: Expand $\ln\left(\frac{x^3\sqrt{x+5}}{x^2+4}\right)$.

The graph of $y = \ln x$:

It can be shown that $\lim_{x\to\infty} \ln x = \infty$ and that $\lim_{x\to 0^+} = -\infty$.

For $x > 0$, $\frac{dy}{dx} = \frac{1}{x} > 0$ so $y = \ln x$ is increasing on $(0,\infty)$.

For $x > 0$, $\frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$ so $y = \ln x$ is concave down on $(0,\infty)$.



Because $\ln 1 = 0$ and $y = \ln x$ is increasing to arbitrarily large values $\left(\lim_{x\to\infty} \ln x = \infty\right)$, the Intermediate Value Theorem guarantees that there is a number *x* such that $\ln x = 1$. That number is called *e*.

$e \approx 2.71828182845904523536$

(*e* is in irrational number—it cannot be written as a decimal that ends or repeats.)

Example 2: Find $\frac{dy}{dx}$ for $y = \ln(2x^5 + 3x)$.

Note: $\frac{d}{dx}\left(\ln u\right) = \frac{1}{u}\frac{du}{dx}$ or, written another way, $\frac{d}{dx}\left(\ln g(x)\right) = \frac{g'(x)}{g(x)}$.

Example 3: Determine $\frac{d}{dx}\left(\ln(\cos x)\right)$.

Example 4: Find the derivative of $f(x) = \frac{1}{\ln x}$.

Example 5: Find the derivative of $f(x) = x^2 \ln x$.

Example 6: Find the derivative of $y = \frac{\ln x}{4x}$.

Example 7: Find the derivative of $g(t) = \ln(7t)$.

Example 8: Determine the derivative of $f(x) = \dfrac{\ln 6x}{(x+4)^5}$.

Logarithmic differentiation:

To differentiate $y = f(x)$:

1. Take the natural logarithm of both sides.

2. Use the laws of logarithms to expand.

3. Differentiate implicitly with respect to *x*.

4. Solve for $\dfrac{dy}{dx}$.

Example 9: Use logarithmic differentiation to find the derivative of

$$y=(x^2+2)^5(2x+1)^3(6x-1)^2.$$

Example 10: Find y' for $y=\dfrac{(x^3+1)^4\sin^2 x}{\sqrt[3]{x}}$.