

5.1: The Natural Logarithmic Function: Differentiation

An algebraic approach to logarithms:

$$\text{Ex: } \log_2 8 = 3 \\ \text{because } 2^3 = 8,$$

Definition: $\log_b x = y$ is equivalent to $b^y = x$.

The functions $f(x) = b^x$ and $g(x) = \log_b x$ are inverses of each other.
 b is called the *base* of the logarithm.

The logarithm of base e is called the *natural logarithm*, which is abbreviated “ln”.

The natural logarithm:

$$\ln x = \log_e x .$$

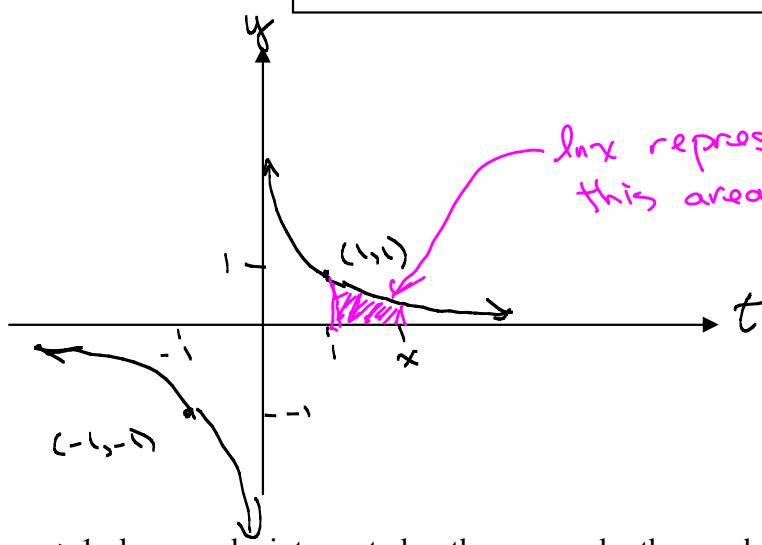
Therefore $\ln x = y$ is equivalent to $e^y = x$ and the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverses of each other.

A calculus approach to the natural logarithm:

The natural logarithm function is defined as

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0 .$$

$$\ln x = \int_1^x \frac{1}{t} dt$$



$$y = \frac{1}{t}$$

For $x > 1$, $\ln x$ can be interpreted as the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to $t = x$.

Note: The integral is not defined for $x < 0$. (*improper integral*)

For $x = 1$, $\ln x = \int_1^1 \frac{1}{t} dt = 0$.

For $x < 1$, $\ln x = \int_1^x \frac{1}{t} dt = -\int_x^1 \frac{1}{t} dt < 0$.

Recall:

The Fundamental Theorem of Calculus, Part II:

Let f be continuous on the interval $[a, b]$. Then the function g defined by

$$g(x) = \int_a^x f(t) dt, \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b) , and $g'(x) = f(x)$.

Apply the Fundamental Theorem of Calculus to the function $f(t) = \frac{1}{t}$.

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$$\frac{d}{dx} \left(\int_1^x \frac{1}{t} dt \right) = \frac{1}{x}$$

This means that $\frac{d}{dx} (\ln x) = \frac{1}{x}$. (because $\ln x = \int_1^x \frac{1}{t} dt$)

The Derivative of the Natural Logarithmic Function

$$\frac{d}{dx} (\ln x) = \frac{1}{x}$$

Laws of Logarithms:

If x and y are positive numbers and r is a rational number, then:

$$1. \ln(xy) = \ln x + \ln y$$

$$2. \ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

Note: This also gives us $\ln\left(\frac{1}{x}\right) = -\ln x$.

$$3. \ln(x^r) = r \ln x$$

Example 1: Expand $\ln\left(\frac{x^3\sqrt{x+5}}{x^2+4}\right)$.

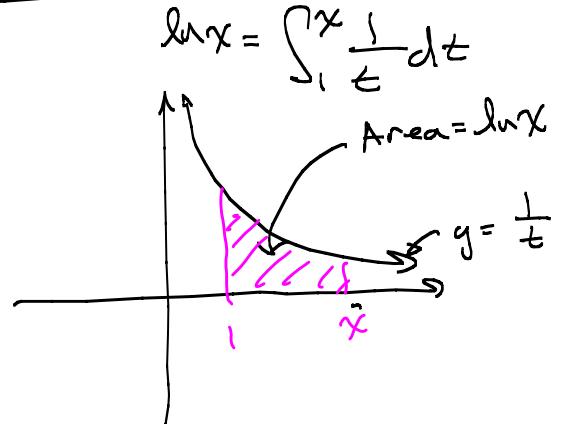
$$\begin{aligned} \ln\left(\frac{x^3(x+5)^{\frac{1}{2}}}{x^2+4}\right) &= \ln\left[x^3(x+5)^{\frac{1}{2}}\right] - \ln(x^2+4) \\ &= \ln x^3 + \ln(x+5)^{\frac{1}{2}} - \ln(x^2+4) \quad \boxed{3\ln x + \frac{1}{2}\ln(x+5) - \ln(x^2+4)} \end{aligned}$$

The graph of $y = \ln x$:

It can be shown that $\lim_{x \rightarrow \infty} \ln x = \infty$ and that $\lim_{x \rightarrow 0^+} \ln x = -\infty$.

For $x > 0$, $\frac{dy}{dx} = \frac{1}{x} > 0$ so $y = \ln x$ is increasing on $(0, \infty)$.

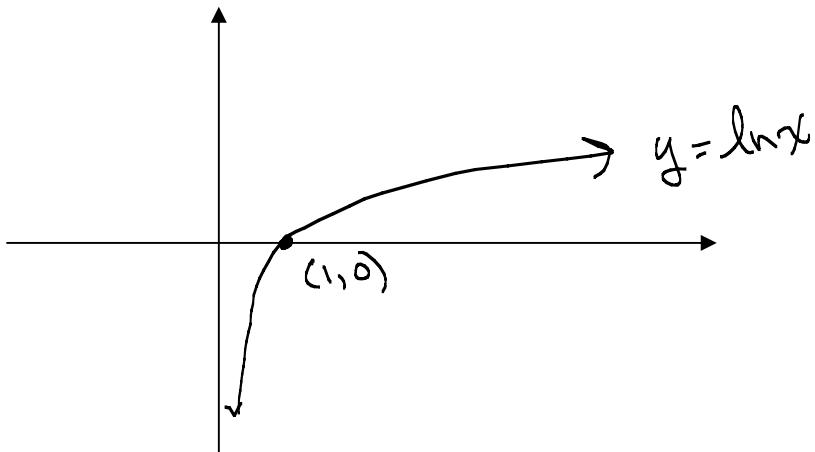
For $x > 0$, $\frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$ so $y = \ln x$ is concave down on $(0, \infty)$.



Already knew that

$$\ln(1) = 0$$

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Because $\ln 1 = 0$ and $y = \ln x$ is increasing to arbitrarily large values ($\lim_{x \rightarrow \infty} \ln x = \infty$), the Intermediate Value Theorem guarantees that there is a number x such that $\ln x = 1$. That number $f(c) = e$ is called e .

$$\begin{array}{l} \ln(b) = f(b) = 100 \\ \ln(1) = f(1) = 0 \end{array}$$

(e is an irrational number—it cannot be written as a decimal that ends or repeats.)

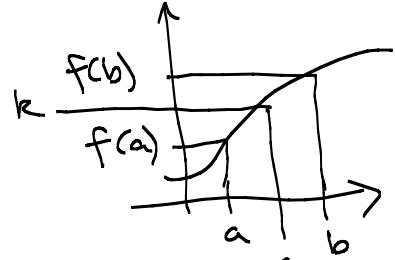
$$\ln e = 1$$

Because $\ln 1 \leq 1 \leq \ln b$

there must be a

$c \in (1, b)$ such that $f(c) = 1$

That c turns out to be e



There is a c such that

Note: $u = 2x^5 + 3x$, so if $f(x) = \ln(2x^5 + 3x)$, then $f(u) = \ln u$ 5.1.4

Example 2: Find $\frac{dy}{dx}$ for $y = \ln(2x^5 + 3x)$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2x^5 + 3x} \cdot \frac{d}{dx}(2x^5 + 3x) \\ &= \frac{1}{2x^5 + 3x} \cdot (10x^4 + 3)\end{aligned}$$

Note: $\frac{d}{dx}(\ln u) = \frac{1}{u} \frac{du}{dx}$ or, written another way, $\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2x^5 + 3x} \cdot (10x^4 + 3) \\ &= \frac{10x^4 + 3}{2x^5 + 3x} \\ &= \boxed{\frac{10x^4 + 3}{2x^5 + 3x}}\end{aligned}$$

Example 3: Determine $\frac{d}{dx}(\ln(\cos x))$.

$$\begin{aligned}\frac{d}{dx}(\ln(\cos x)) &= \frac{1}{\cos x} \frac{d}{dx}(\cos x) \\ &= \frac{1}{\cos x} (-\sin x) \\ &= \boxed{-\tan x}\end{aligned}$$

Example 4: Find the derivative of $f(x) = \frac{1}{\ln x}$.

$$\begin{aligned}y &= \cos(2x^5 + 3x) \\ \frac{dy}{dx} &= -\sin(2x^5 + 3x) \frac{d}{dx}(2x^5 + 3x) \\ &= -\sin(2x^5 + 3x)(10x^4 + 3) \\ &= \boxed{-(10x^4 + 3)\sin(2x^5 + 3x)}\end{aligned}$$

$$\begin{aligned}f(x) &= (\ln x)^{-1} \\ f'(x) &= -1(\ln x)^{-2} \frac{d}{dx}(\ln x) = -1(\ln x)^{-2} \left(\frac{1}{x}\right) = \boxed{-\frac{1}{x(\ln x)^2}}\end{aligned}$$

Example 5: Find the derivative of $f(x) = x^2 \ln x$. Need Product Rule:

$$\begin{aligned}f'(x) &= x^2 \frac{d}{dx}(\ln x) + (\ln x) \frac{d}{dx}(x^2) = x^2 \left(\frac{1}{x}\right) + (\ln x)(2x) \\ &= \boxed{x + 2x \ln x} = \boxed{x(1 + 2 \ln x)}\end{aligned}$$

Example 6: Find the derivative of $y = \frac{\ln x}{4x}$.

$$\text{Quotient Rule: } \frac{dy}{dx} = \frac{(4x) \frac{d}{dx}(\ln x) - (\ln x) \frac{d}{dx}(4x)}{(4x)^2}$$

$$\begin{aligned}&= \frac{4x \left(\frac{1}{x}\right) - (\ln x)(4)}{16x^2} = \frac{4 - 4 \ln x}{16x^2} = \frac{\cancel{4}(1 - \ln x)}{\cancel{16}x^2} \\ &= \boxed{\frac{1 - \ln x}{4x^2}}\end{aligned}$$

Example 7: Find the derivative of $g(t) = \ln(7t)$.

$$g'(t) = \frac{1}{7t} \frac{d}{dt}(7t) = \frac{1}{7t} (7) = \boxed{\frac{1}{t}}$$

Note: same derivative as $f(t) = \ln(t)$

Note: $g(t) = \ln(7t) = \ln(7) + \ln(t)$

Graph of g : Start with graph of $f(t) = \ln(t)$ and shift it vertically by $\ln 7$.

Example 8: Determine the derivative of $f(x) = \frac{\ln 6x}{(x+4)^5}$.

$$\begin{aligned} f'(x) &= \frac{(x+4)^5 \frac{d}{dx}(\ln(6x)) - (\ln(6x)) \frac{d}{dx}(x+4)^5}{(x+4)^{10}} \\ &= \frac{(x+4)^5 \left(\frac{1}{6x}\right)(6) - (\ln(6x))(5)(x+4)^4(1)}{(x+4)^{10}} \\ &= \frac{(x+4)^5 \left(\frac{1}{x}\right) - 5(x+4)^4 \ln(6x)}{(x+4)^{10}} \\ &= \frac{(x+4)^4 \left[(x+4)\left(\frac{1}{x}\right) - 5 \ln(6x)\right]}{(x+4)^{10}} = \frac{(x+4)\left(\frac{1}{x}\right) - 5 \ln(6x)}{(x+4)^6} \left(\frac{x}{x}\right) \\ &= \boxed{\frac{x+4 - 5x \ln(6x)}{(x+4)^6}} \end{aligned}$$

Logarithmic differentiation:

To differentiate $y = f(x)$:

1. Take the natural logarithm of both sides.

2. Use the laws of logarithms to expand.

3. Differentiate implicitly with respect to x .

4. Solve for $\frac{dy}{dx}$.

Example 9: Use logarithmic differentiation to find the derivative of

$$y = (x^2 + 2)^5 (2x+1)^3 (6x-1)^2.$$

Take \ln of both sides: $\ln y = \ln [(x^2 + 2)^5 (2x+1)^3 (6x-1)^2]$

Apply log properties: $\ln y = 5\ln(x^2 + 2) + 3\ln(2x+1) + 2\ln(6x-1)$

Implicit diff: $\frac{d}{dx}(\ln y) = \frac{d}{dx}[5\ln(x^2 + 2) + 3\ln(2x+1) + 2\ln(6x-1)]$

$\frac{1}{y} \cdot \frac{dy}{dx} = 5\left(\frac{1}{x^2+2}\right)(2x) + 3\left(\frac{1}{2x+1}\right)(2) + 2\left(\frac{1}{6x-1}\right)(6)$

$\frac{dy}{dx} = y \left[\frac{10x}{x^2+2} + \frac{6}{2x+1} + \frac{12}{6x-1} \right] \Rightarrow \frac{dy}{dx} = (x^2+2)^5 (2x+1)^3 (6x-1)^2 \left[\frac{10x}{x^2+2} + \frac{6}{2x+1} + \frac{12}{6x-1} \right]$

Example 10: Find y' for $y = \frac{(x^3+1)^4 \sin^2 x}{\sqrt[3]{x}}$.

$$y = \frac{(x^3+1)^4 (\sin x)^2}{x^{1/3}}$$

$$\ln y = \ln \frac{(x^3+1)^4 (\sin x)^2}{x^{1/3}}$$

$$\ln y = 4\ln(x^3+1) + 2\ln(\sin x) - \frac{1}{3}\ln x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 4\left(\frac{1}{x^3+1}\right)(3x^2) + 2\left(\frac{1}{\sin x}\right)(\cos x) - \frac{1}{3} \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y \left[\frac{12x^2}{x^3+1} + 2\cot x - \frac{1}{3x} \right]$$

$$= \frac{(x^3+1)^4 \sin^2 x}{\sqrt[3]{x}} \left[\frac{12x^2}{x^3+1} + 2\cot x - \frac{1}{3x} \right]$$

Ex. 4

$$\text{Ex 4} \begin{cases} f(x) = \frac{1}{\ln(\cos x)} \\ f(x) = (\ln(\cos x))^{-1} \end{cases}$$

$$f'(x) = -1 (\ln(\cos x))^{-2} \frac{d}{dx} (\ln(\cos x)) \\ = -1 (\ln(\cos x))^{-2} \frac{1}{\cos x} \frac{d}{dx} (\cos x)$$

$$= -\frac{1}{(\ln(\cos x))^2 \cos x} (-\sin x) = \frac{\sin x}{(\ln(\cos x))^2 \cos x}$$

$$= \boxed{\frac{\tan x}{(\ln(\cos x))^2}}$$

Ex 6: $y = \frac{\ln x}{4x}$

using product Rule: $y = (\ln x)(4x)^{-1}$

$$\begin{aligned} \frac{dy}{dx} &= (\ln x) \frac{d}{dx} (4x)^{-1} + (4x)^{-1} \frac{d}{dx} (\ln x) \\ &= (\ln x)(-1)(4x)^{-2}(4) + (4x)^{-1}\left(\frac{1}{x}\right) \\ &= -\frac{4\ln x}{(4x)^2} + \frac{1}{(4x)x} = -\frac{4\ln x}{16x^2} + \frac{1}{4x^2} \\ &= -\frac{\ln x}{4x^2} + \frac{1}{4x^2} = \boxed{\frac{1-\ln x}{4x^2}} \end{aligned}$$