

5.2: The Natural Logarithmic Function: Integration

Using the derivative of the natural logarithmic function to obtain an antiderivative:

Example 1: Find the derivative of $g(x) = \ln|x|$.

Rewrite $g(x)$ without the absolute values.

$$g(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases} \quad (\text{undefined for } x=0)$$

$$g'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases}$$

so $g'(x) = \frac{1}{x}$ for all x in its domain
(so all x except 0)

Note that $f(x) = \ln x$ has the same derivative as $g(x) = \ln|x|$.

Therefore $\frac{d}{dx} \ln|x| = \frac{1}{x}$. This means that $f(x) = \ln|x|$ is an antiderivative of $F(x) = \frac{1}{x}$.

$$\int \frac{1}{x} dx = \ln|x| + C$$

Recall: The power rule for integrals $\int x^n dx = \frac{x^{n+1}}{n+1}$ had a restriction: $n \neq -1$. Now we can handle this case.

$$n \neq -1$$

$$\text{Check: } \frac{d}{dx} \left(\frac{1}{3} \ln(x^3 + 4) \right) = \frac{1}{3} \left(\frac{1}{x^3 + 4} \right) (3x^2)$$

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$$= \frac{x^2}{x^3 + 4} \quad \text{OK}$$

Example 2: Determine $\int \frac{x^2}{x^3 + 4} dx$.

$$\begin{aligned} \int x^2 \left(\frac{1}{x^3 + 4} \right) dx &= \frac{1}{3} \int \frac{1}{u} du \\ &= \frac{1}{3} \ln|u| + C \\ &= \boxed{\frac{1}{3} \ln|x^3 + 4| + C} \end{aligned}$$

$$\left. \begin{aligned} u &= x^3 + 4 \\ \frac{du}{dx} &= 3x^2 \\ du &= 3x^2 dx \\ \frac{1}{3} du &= x^2 dx \end{aligned} \right\}$$

Example 3: Determine $\int \frac{7}{2-5x} dx$.

$$= 7 \int \frac{1}{2-5x} dx$$

$\frac{1}{u}$ $-\frac{1}{5} du$

$$\begin{aligned} &= 7 \left(-\frac{1}{5} \right) \int \frac{1}{u} du = -\frac{7}{5} \ln|u| + C \\ &= \boxed{-\frac{7}{5} \ln|2-5x| + C} \end{aligned}$$

$$u = 2 - 5x$$

$$\frac{du}{dx} = -5$$

$$du = -5 dx$$

$$-\frac{1}{5} du = dx$$

Example 4: Determine $\int_2^5 \frac{1}{3x} dx$.

$$\begin{aligned} \int_2^5 \frac{1}{3x} dx &= \frac{1}{3} \int_2^5 \frac{1}{x} dx = \frac{1}{3} \ln|x| \Big|_2^5 \\ &= \frac{1}{3} \ln|5| - \frac{1}{3} \ln|2| = \boxed{\frac{1}{3} (\ln 5 - \ln 2)} \end{aligned}$$

or use substitution:

$$\left. \begin{aligned} u &= 3x \\ du &= 3dx \end{aligned} \right\} \Rightarrow \frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln|u| \Big|_{x=2}^{x=5} = \frac{1}{3} \ln|3x| \Big|_{x=2}^{x=5}$$

$$= \boxed{\frac{1}{3} \ln\left(\frac{5}{2}\right)}$$

Example 5: Determine $\int \frac{x^7 - x + 3x^4}{x^5} dx$.

$$\begin{aligned} &= \frac{1}{3} \ln|3 \cdot 5| - \frac{1}{3} \ln|3 \cdot 2| \\ &= \frac{1}{3} (\ln 15 - \ln 6) = \frac{1}{3} \ln\left(\frac{15}{6}\right) \\ &= \boxed{\frac{1}{3} \ln\left(\frac{5}{2}\right)} \end{aligned}$$

$$\int \left(\frac{x^7}{x^5} - \frac{x}{x^5} + \frac{3x^4}{x^5} \right) dx = \int \left(x^2 - x^{-4} + \frac{3}{x} \right) dx$$

$$= \frac{x^3}{3} - \frac{x^{-3}}{-3} + 3 \ln|x| + C$$

$$= \boxed{\frac{x^3}{3} + \frac{1}{3x^3} + 3 \ln|x| + C}$$

Example 6: Find $\int \frac{(\ln x)^4}{x} dx$.

$$\begin{aligned} \int (\underbrace{(\ln x)^4}_{u^4} \underbrace{\left(\frac{1}{x}\right) dx}_{du}) &= \int u^4 du = \frac{u^5}{5} + C \\ &= \boxed{\frac{(\ln x)^5}{5} + C} \end{aligned}$$

$$u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{1}{x} dx$$

$$\begin{aligned} \text{Check: } \frac{d}{dx} \left(\frac{1}{5} (\ln x)^5 \right) \\ = \cancel{\frac{1}{5}} (5) (\ln x)^4 \left(\frac{1}{x} \right) \end{aligned}$$

Example 7: Find $\int \frac{\ln(3x)}{x} dx$.

$$\begin{aligned} &= \int (\underbrace{\ln(3x)}_{u} \underbrace{\left(\frac{1}{x}\right) dx}_{du}) \\ &= \int u du = \frac{u^2}{2} + C = \boxed{\frac{(\ln(3x))^2}{2} + C} \end{aligned}$$

$$\begin{aligned} u &= \ln(3x) \\ \frac{du}{dx} &= \frac{1}{3x} (3) \end{aligned}$$

$$\begin{aligned} \frac{du}{dx} &= \frac{1}{3x} \\ &= \frac{1}{x} \\ du &= \frac{1}{x} dx \end{aligned}$$

$$\begin{aligned} \text{Try } u &= 3x \\ \frac{du}{dx} &= 3 \\ du &= 3 dx \\ \text{No.} \end{aligned}$$

Example 8: Find $\int \frac{x}{x^2 - 8} dx$.

Example 9: Find $\int \frac{4x^2 - 5x - 12}{x^2 - 3} dx$.

$$\text{Integral} = \int \left(4 - \frac{5x}{x^2 - 3} \right) dx$$

$$= \int 4 dx - 5 \int \frac{x}{x^2 - 3} dx$$

$$= 4x - 5 \left(\frac{1}{2} \right) \int \frac{1}{u} du$$

$$= 4x - \frac{5}{2} \ln|u| + C = \boxed{4x - \frac{5}{2} \ln|x^2 - 3| + C}$$

Long Division:

$$\begin{array}{r} 4 \\ x^2 - 3 \overline{)4x^2 - 5x - 12} \\ - (4x^2 - 12) \\ \hline -5x + 0 \end{array}$$

$$u = x^2 - 3$$

$$\frac{du}{dx} = 2x$$

$$du = 2x dx$$

$$\frac{1}{2} du = x dx$$

$$\frac{4x^2}{2x} - \frac{x}{2x}$$

$$\downarrow \quad \uparrow$$

$$2x - \frac{1}{2}$$

$$2x - 3 \int 4x^2 - 7x + 1$$

$$-(4x^2 - 6x)$$

$$\frac{-x + 1}{-(-x + 3/2)}$$

$$-\frac{1}{2}$$

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Example 10: Find $\int \frac{4x^2 - 7x + 1}{2x-3} dx$.

$$\text{Integrand} = \int \left(2x - \frac{1}{2} - \frac{1/2}{2x-3}\right) dx$$

$$= \int 2x dx - \int \frac{1}{2} dx - \frac{1}{2} \int \frac{1}{2x-3} dx$$

$$= \frac{2x^2}{2} - \frac{1}{2}x - \frac{1}{2} \int \frac{1}{u} \left(\frac{1}{2}\right) du$$

$$= x^2 - \frac{1}{2}x - \frac{1}{4} \int \frac{1}{u} du$$

$$= x^2 - \frac{1}{2}x - \frac{1}{4} \ln|u| + C = \boxed{x^2 - \frac{1}{2}x - \frac{1}{4} \ln|2x-3| + C}$$

$$\begin{aligned} u &= 2x-3 \\ \frac{du}{dx} &= 2 \\ du &= 2 dx \\ \frac{1}{2} du &= dx \end{aligned}$$

Integrating the remaining trigonometric functions:

Example 11: Determine $\int \tan x dx$.

$$\begin{aligned} \int \tan x dx &= \int \frac{\sin x}{\cos x} dx = \int \sin x \left(\frac{1}{\cos x}\right) dx \\ &= - \int \frac{1}{u} du \quad \text{where } u = \cos x \\ &= - \ln|u| + C = \boxed{-\ln|\cos x| + C} \end{aligned}$$

$$\begin{aligned} \text{Try } u &= \sin x \\ du &= \cos x dx \\ \text{don't have it} \end{aligned}$$

$$\begin{aligned} u &= \cos x \\ \frac{du}{dx} &= -\sin x \\ du &= -\sin x dx \\ -du &= \sin x dx \end{aligned}$$

Example 12: Determine $\int \cot x dx$.

$$\begin{aligned} \int \cot x dx &= \int \frac{\cos x}{\sin x} dx = \int \cos x \cdot \frac{1}{\sin x} dx \\ &\quad \text{where } u = \sin x \\ &= \int \frac{1}{u} du = \ln|u| + C \end{aligned}$$

$$\begin{aligned} u &= \sin x \\ \frac{du}{dx} &= \cos x \\ du &= \cos x dx \end{aligned}$$

$$\begin{aligned} &= \ln|\sin x| + C \end{aligned}$$

$$\begin{aligned} \text{check: } \frac{d}{dx} (\ln(\sin x)) &= \frac{1}{\sin x} \frac{d}{dx} (\sin x) \\ &= \frac{1}{\sin x} \cos x = \cot x \end{aligned}$$

$$\int \sec x \, dx = \int \frac{1}{\cos x} \, dx$$

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$$u = \cos x$$

nothing for du
doesn't work

Example 13: Determine $\int \sec x \, dx$.

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{1}{u} \, du = \ln|u| + C$$

$$= \boxed{\ln|\sec x + \tan x| + C}$$

$$u = \sec x + \tan x$$

$$\frac{du}{dx} = \sec x \tan x + \sec^2 x$$

$$du = (\sec x \tan x + \sec^2 x) \, dx$$

Example 14: Determine $\int \csc x \, dx$.

$$\int \csc x \, dx = \int \csc x \cdot \frac{\csc x + \cot x}{\csc x + \cot x} \, dx$$

$$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx$$

$$= \int \frac{1}{\csc x + \cot x} \cdot \frac{\csc^2 x + \csc x \cot x}{\csc^2 x + \csc x \cot x} \, dx$$

$$u = \csc x + \cot x$$

$$\frac{du}{dx} = -\csc x \cot x - \csc^2 x$$

$$-du = \csc x \cot x + \csc^2 x \, dx$$

$$= - \int \frac{1}{u} \, du$$

$$= -\ln|u| + C$$

$$= \boxed{-\ln|\csc x + \cot x| + C}$$