

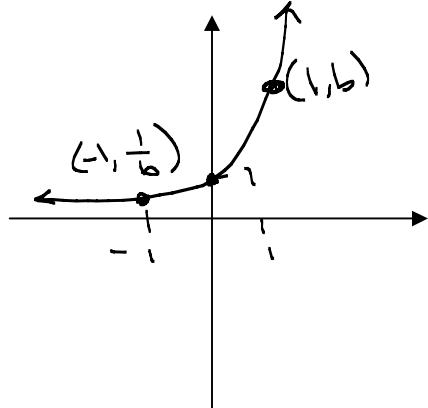
5.4: Exponential Functions: Differentiation and Integration

Short Review:

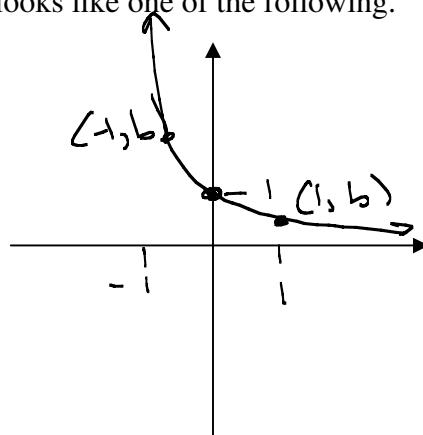
An *exponential* function takes the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

For any exponential function $f(x) = b^x$, the graph looks like one of the following.

$$\begin{array}{|c|l|} \hline x & y = b^x \\ \hline 0 & b^0 = 1 \\ 1 & b^1 = b \\ -1 & b^{-1} = \frac{1}{b} \\ \hline \end{array}$$



$$b > 1$$



$$0 < b < 1$$

Notice:

- Domain is $(-\infty, \infty)$.
- Range is $(0, \infty)$.
- Horizontal asymptote is $y = 0$.
- Always passes through the points $(0, 1)$, $(1, b)$, $(-1, \frac{1}{b})$

The natural exponential function:

The number e can be defined in several ways.

One definition of the number e :

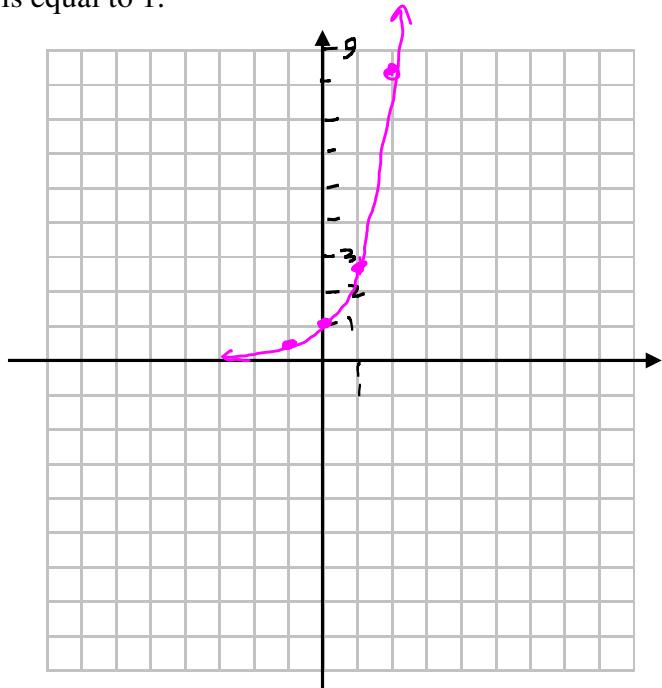
e is the number such that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$

$$e \approx 2.718281828459$$

The slope of the tangent line at the point $(0,1)$ is equal to 1.

The graph of $f(x) = e^x$:

$$\begin{array}{|c|c|} \hline x & e^x \\ \hline -1 & e^{-1} = \frac{1}{e} \approx \frac{1}{3} \\ 0 & e^0 = 1 \\ 1 & e^1 \approx 2.7 \\ 2 & e^2 \approx 9 \text{ (a little less than 9)} \\ \hline \end{array}$$



Another definition of the number e :

$$e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \quad \text{or, equivalently, } e = \lim_{x \rightarrow 0} (1+x)^{1/x}$$

$$e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

Let $n = \frac{1}{x}$. $\Rightarrow \begin{cases} n \rightarrow \infty \\ x = \frac{1}{n} \end{cases}$

Derivatives of exponential functions:

$$\frac{d}{dx}(e^x) = e^x$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x &= \lim_{n \rightarrow 0} \left(1 + n\right)^{1/n} \\ &= \lim_{x \rightarrow 0} (1+x)^{1/x} \end{aligned}$$

Example 1: Find the derivative of $f(x) = -7e^x$.

$$f'(x) = -7e^x$$

Note: $\frac{d}{dx}(\sin(-7x)) = \cos(-7x) \frac{d}{dx}(-7x)$

$$= (\cos(-7x))(-7) = \boxed{-7 \cos(-7x)}$$

5.4.3

Example 2: Find the derivative of $f(x) = 5\sqrt{e^x + 7}$.

$$f(x) = 5(e^x + 7)^{1/2}$$

$$f'(x) = 5\left(\frac{1}{2}\right)(e^x + 7)^{-1/2} \frac{d}{dx}(e^x + 7) = \frac{5}{2} (e^x + 7)^{-1/2} (e^x)$$

$$= \boxed{\frac{5e^x}{2\sqrt{e^x + 7}}}$$

Example 3: Find the derivative of $f(x) = e^x \sin x$.

$$f'(x) = e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$= \boxed{e^x \cos x + e^x \sin x} = e^x (\cos x + \sin x)$$

Example 4: Find the derivative of $g(x) = e^{-7x} + 2x^3 - 4e$.

$$g'(x) = e^{-7x} \frac{d}{dx}(-7x) + 2(3x^2) + 0$$

$$= e^{-7x} (-7) + 6x^2 = \boxed{-7e^{-7x} + 6x^2}$$

Note:
 $\frac{d}{dx}(e^x) = 0$
 $(e, e^x, e^2 \text{ are constants})$

Example 5: Find the derivative of $y = e^{x^2+4x}$.

$$\frac{dy}{dx} = e^{x^2+4x} \frac{d}{dx}(x^2+4x) = \boxed{e^{x^2+4x} (2x+4)}$$

(chain rule)

Example 6: Find the derivative of $f(x) = \cos(e^x - x)$.

$$f'(x) = -\sin(e^x - x) \frac{d}{dx}(e^x - x)$$

$$= \boxed{-\sin(e^x - x)(e^x - 1)} = \boxed{-(e^x - 1)\sin(e^x - x)}$$

$$= \boxed{-e^x \sin(e^x - x) + \sin(e^x - x)}$$

Example 7: Find the equation of the tangent line to the graph of $f(x) = (e^x + 2)^2$ at the point $(0, 9)$.

$$f'(x) = 2(e^x + 2) \frac{d}{dx}(e^x + 2)$$

$$= 2(e^x + 2)(e^x + 0) = 2e^x (e^x + 2)$$

slope: $m = f'(0) = 2e^0(e^0 + 2) = 2(1)(1+2) = 2(3) = 6$

check/find point: $f(0) = (e^0 + 2)^2 = (1+2)^2 = 3^2 = 9 \checkmark$

$$y - y_1 = m(x - x_1)$$

or $y = mx + b$

$$y - 9 = 6(x - 0)$$

$$y - 9 = 6x$$

$$y = 6x + 9$$

$$y = \boxed{6x + 9}$$

Integration of exponential functions:

$$\int e^x dx = e^x + C$$

Example 8: Determine $\int (x^2 - 5e^x) dx$

$$\int (x^2 - 5e^x) dx = \boxed{\frac{x^3}{3} - 5e^x + C}$$

Example 9: Find $\int e^{5t} dt$.

$$\int e^{5t} dt = \frac{1}{5} \int e^u du$$

$$= \frac{1}{5} e^u + C = \boxed{\frac{1}{5} e^{5t} + C}$$

$$u = 5t$$

$$\begin{aligned} \frac{du}{dt} &= 5 \\ du &= 5 dt \\ \frac{1}{5} du &= dt \end{aligned}$$

Example 10: Find $\int_1^3 e^{2x-3} dx$.

$$\int_1^3 e^{2x-3} dx = \frac{1}{2} \int_{u=-1}^{u=3} e^u du$$

$$= \frac{1}{2} e^u \Big|_{u=-1}^u = \boxed{\frac{1}{2} [e^3 - e^{-1}]}$$

$$\begin{aligned} \text{Check: } \frac{d}{dt} \left(\frac{1}{2} e^{5t} \right) &= \frac{1}{2} \cdot e^{5t} \cdot 5 = e^{5t} \\ \checkmark & \end{aligned}$$

or (without changing limits of integration)

$$\int_1^3 e^{2x-3} dx = \frac{1}{2} \int_{x=1}^{x=3} e^u du = \frac{1}{2} e^u \Big|_{x=1}^{x=3}$$

$$= \frac{1}{2} e^{2(3)-3} \Big|_{x=1}^{x=3} = \frac{1}{2} \left[e^{2(3)-3} - e^{2(1)-3} \right]$$

$$u = 2x - 3$$

$$\begin{aligned} \frac{du}{dx} &= 2 \\ du &= 2 dx \\ \frac{1}{2} du &= dx \end{aligned}$$

$$\begin{aligned} x = 1 \Rightarrow u &= 2(1) - 3 = -1 \\ x = 3 \Rightarrow u &= 2(3) - 3 \\ &= 6 - 3 = 3 \end{aligned}$$

$$\boxed{\frac{1}{2} [e^3 - e^{-1}]}$$

Example 11: Find $\int te^{t^2} dt$.

$$\begin{aligned} \int te^{t^2} dt &= \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + C \\ &= \boxed{\frac{1}{2} e^{t^2} + C} \end{aligned}$$

$$u = t^2$$

$$\frac{du}{dt} = 2t$$

$$du = 2t dt$$

$$\frac{1}{2} du = t dt$$

Example 12: Determine $\int \frac{e^x}{\sqrt[3]{e^x + 1}} dx$.

$$\begin{aligned} \int e^x (e^x + 1)^{-\frac{1}{3}} dx &= \int u^{-\frac{1}{3}} du \\ &= \frac{u^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C = \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C = \frac{3u^{\frac{2}{3}}}{2} + C \\ &= \boxed{\frac{3}{2} (e^x + 1)^{\frac{2}{3}} + C} \end{aligned}$$

$$u = e^x + 1$$

$$\frac{du}{dx} = e^x$$

$$du = e^x dx$$

Example 13: Determine $\int \frac{e^x - e^{-x}}{e^{3x}} dx$

$$\begin{aligned} \int \left(\frac{e^x}{e^{3x}} - \frac{e^{-x}}{e^{3x}} \right) dx &= \int (e^{x-3x} - e^{-x-3x}) dx \\ &= \int (e^{-2x} - e^{-4x}) dx = \int e^{-2x} dx - \int e^{-4x} dx \\ &= -\frac{1}{2} \int e^u du - \left(-\frac{1}{4} \right) \int e^{u_2} du_2 \\ &= -\frac{1}{2} e^u + \frac{1}{4} e^{u_2} + C \\ &= \boxed{-\frac{1}{2} e^{-2x} + \frac{1}{4} e^{-4x} + C} \end{aligned}$$

$$\begin{aligned} \text{1st one:} \\ u &= -2x \\ du &= -2 dx \\ -\frac{1}{2} du &= dx \end{aligned}$$

2nd one:

$$\begin{aligned} u_2 &= -4x \\ du_2 &= -4 dx \\ -\frac{1}{4} du_2 &= dx \end{aligned}$$