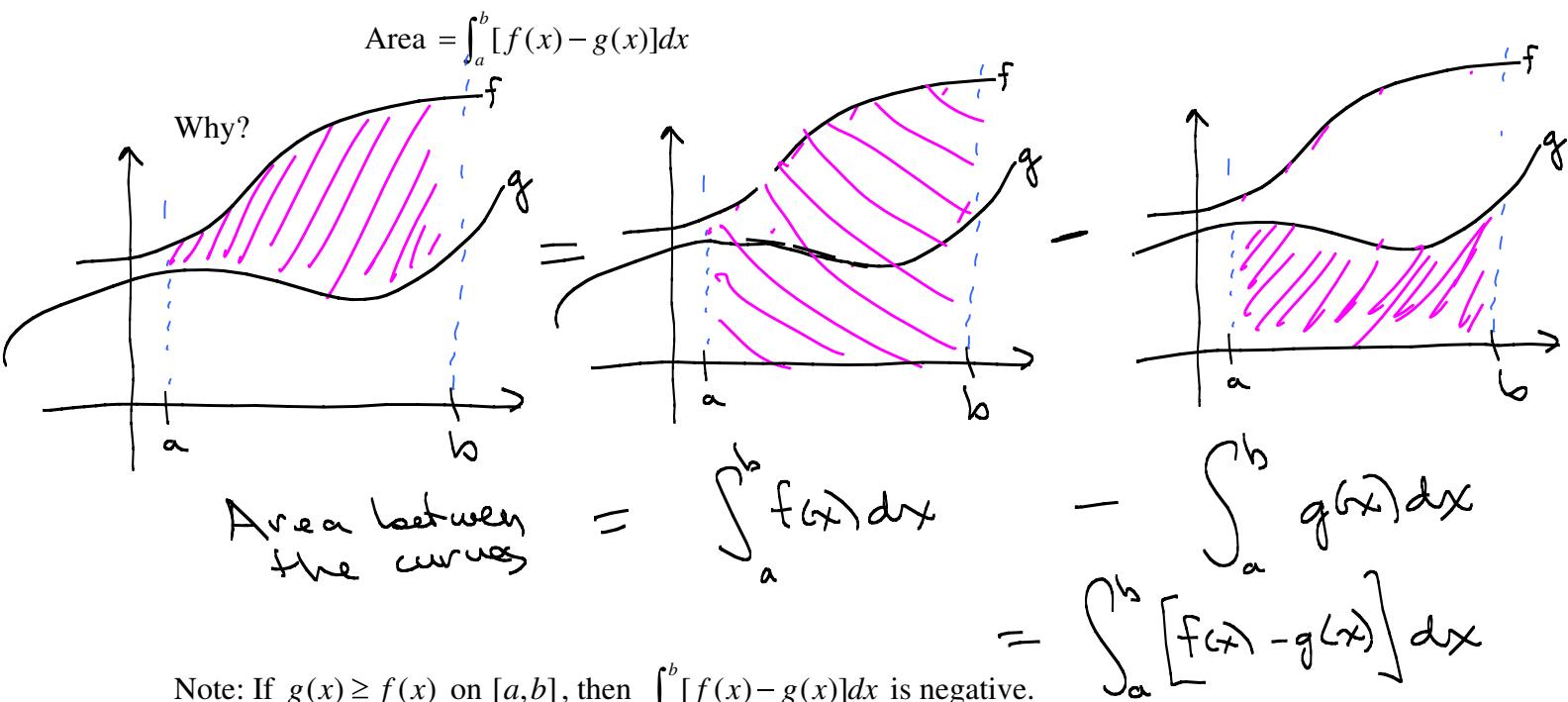


7.1: Area of a Region Between Two Curves

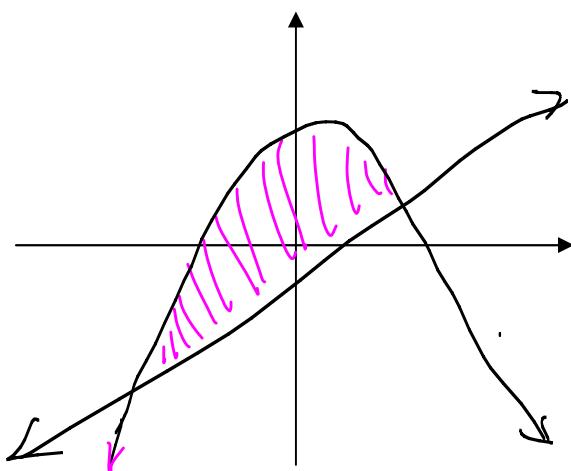
Because the definite integral represents the “net” area under a curve, we can use integration to find the area between curves.

If f and g are continuous and $f(x) \geq g(x)$ on $[a, b]$, then the area between $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ is given by

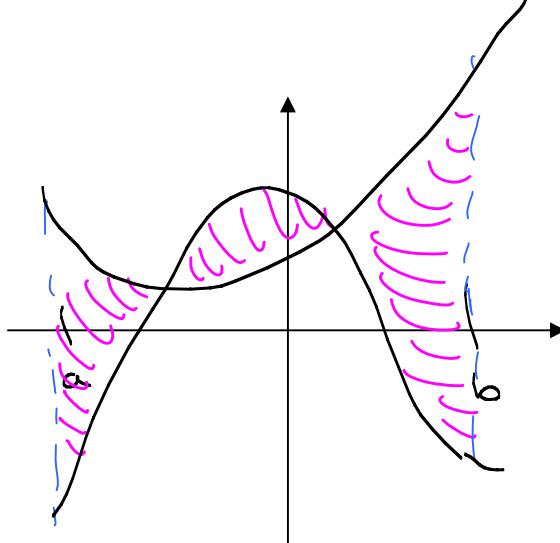


Note: If $g(x) \geq f(x)$ on $[a, b]$, then $\int_a^b [f(x) - g(x)] dx$ is negative.

Some different types of area scenarios:



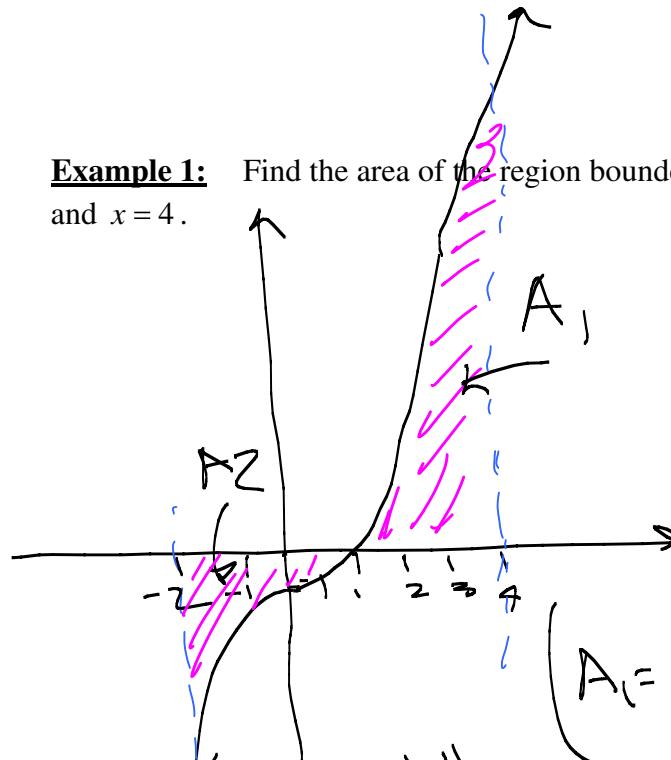
Find area enclosed
by 2 curves



Area is enclosed between
2 curves and between
2 lines (endpoints)

7.1.2

Example 1: Find the area of the region bounded by $f(x) = x^3 - 1$ and the lines $y = 0$, $x = -2$, and $x = 4$.



$$A_1 = \int_{-2}^1 (x^3 - 1) dx = \left(\frac{x^4}{4} - x \right) \Big|_{-2}^1$$

$$= \left(\frac{1}{4} - 1 \right) - \left(\frac{(-2)^4}{4} - (-2) \right)$$

$$= \frac{1}{4} - 1 - 1 - 2 = \frac{1}{4} - 1 - \frac{28}{4} = \frac{-27}{4}$$

$$\text{Area} = |A_1| + |A_2| = \frac{243}{4} + \frac{27}{4} = \frac{270}{4} = 67.50$$

Find intersection pt. of $y = x^3 - 1$ and $y = 0$

$$x^3 - 1 = 0$$

$$x^3 = 1 \Rightarrow x = \sqrt[3]{1}$$

$x = 1$ so the

intersection point
is $(1, 0)$

$$A_1 = \int_{-2}^1 (x^3 - 1) dx = \left(\frac{x^4}{4} - x \right) \Big|_{-2}^1$$

$$= \left(\frac{1}{4} - 1 \right) - \left(\frac{(-2)^4}{4} - (-2) \right)$$

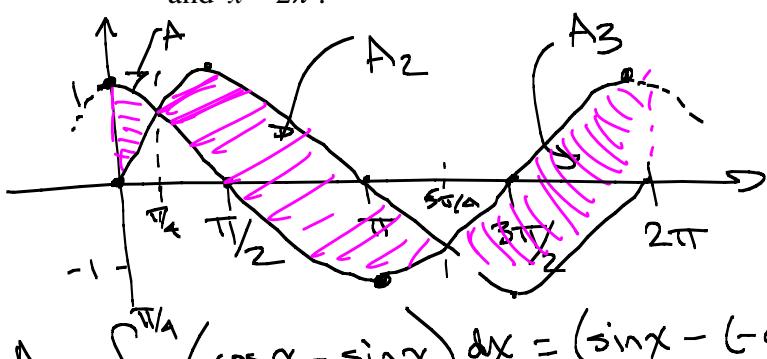
$$= \frac{1}{4} - 1 - 1 - 2 = \frac{1}{4} - 1 - \frac{28}{4} = \frac{-27}{4}$$

$$= (64 - 4) - \frac{1}{4} + 1$$

$$60 - \frac{1}{4} + 1 = \frac{243}{4} - \frac{1}{4} = \frac{242}{4} = 60.50$$

$$\frac{4\sqrt[3]{270}}{4} = \frac{27}{4}$$

Example 2: Find the area of the region bounded by $y = \cos x$, $y = \sin x$, and the lines $x = 0$, and $x = 2\pi$.



$$A_1 = \int_0^{\pi/4} (\cos x - \sin x) dx = (\sin x - (-\cos x)) \Big|_0^{\pi/4} = (\sin x + \cos x) \Big|_0^{\pi/4}$$

$$= \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0 + \cos 0) = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) - (0 + 1) = \frac{2\sqrt{2}}{2} - 1$$

$$A_2 = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx = (-\cos x - \sin x) \Big|_{\pi/4}^{5\pi/4} = \left(-\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4} \right) - \left(-\cos \frac{\pi}{4} - \sin \frac{\pi}{4} \right)$$

$$= \left(-\left(-\frac{\sqrt{2}}{2} \right) - \left(-\frac{\sqrt{2}}{2} \right) \right) + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

$$A_3 = \int_{5\pi/4}^{2\pi} (\cos x - \sin x) dx = (\sin x + \cos x) \Big|_{5\pi/4}^{2\pi} = (\sin 2\pi + \cos 2\pi) - (\sin \frac{5\pi}{4} + \cos \frac{5\pi}{4})$$

$$= 0 + 1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right)$$

next page

Find intersection pts. (really just need the x-values)

Set $\cos x = \sin x$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$= \left(\sin x + \cos x \right) \Big|_0^{\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) - (0 + 1) = \frac{2\sqrt{2}}{2} - 1$$

$$= \sqrt{2} - 1$$

$$= \left(-\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4} \right) - \left(-\cos \frac{\pi}{4} - \sin \frac{\pi}{4} \right)$$

$$= \left(-\left(-\frac{\sqrt{2}}{2} \right) - \left(-\frac{\sqrt{2}}{2} \right) \right) + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

$$A_3 = 1 + \frac{2\sqrt{2}}{2} = 1 + \sqrt{2}$$

$$\text{Area} = |A_1| + |A_2| + |A_3| = \sqrt{2}-1 + 2\sqrt{2} + 1 + \sqrt{2} = \boxed{4\sqrt{2}}$$

7.1.3

Example 3: Find the area of the region completely enclosed by the graphs of $y = x^2 + 1$ and $y = 2x + 9$.

Find intersections: Set the eqns equal:

$$x^2 + 1 = 2x + 9$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = 4, x = -2$$

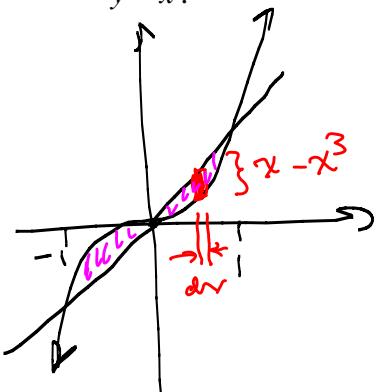
$$\text{Area} = \int_{-2}^4 (2x+9 - (x^2+1)) dx = \int_{-2}^4 (2x+9 - x^2 - 1) dx$$

$$= \int_{-2}^4 (2x+8-x^2) dx = \left(\frac{2x^2}{2} + 8x - \frac{x^3}{3} \right) \Big|_{-2}^4 = \left(x^2 + 8x - \frac{x^3}{3} \right) \Big|_{-2}^4$$

$$= (4^2 + 8(4) - \frac{4^3}{3}) - ((-2)^2 + 8(-2) - \frac{(-2)^3}{3}) = 16 + 32 - \frac{64}{3} - 4 + 16 - \frac{8}{3}$$

Example 4: Find the area of the region completely enclosed by the graphs of $y = x^3$ and $y = x$.

$$y = x.$$



Find intersections:

$$x^3 = x$$

$$x^3 - x = 0$$

$$x(x^2 - 1) = 0$$

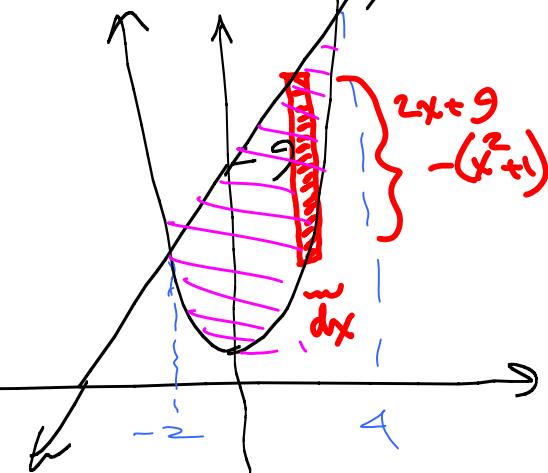
$$x(x+1)(x-1) = 0$$

$$x = 0, -1, 1$$

From symmetry, total area = $2 \int_0^1 (x - x^3) dx$

$$= 2 \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1 = 2 \left[\frac{1^2}{2} - \frac{1^4}{4} \right] - \left(\frac{0^2}{2} - \frac{0^4}{4} \right) = 2 \left(\frac{1}{2} - \frac{1}{4} \right)$$

$$2 \left(\frac{1}{4} \right) = \boxed{\frac{1}{2}}$$



7.1.4

Example 5: Find the area of the region completely enclosed by the graphs of $x = y^2$ and

$$x = 4.$$

$$\text{Area} = \int_{-2}^2 (4 - y^2) dy \quad \begin{matrix} \text{height} \\ \text{width} \end{matrix}$$

$$= 2 \int_0^2 (4 - y^2) dy \quad [\text{from symmetry}]$$

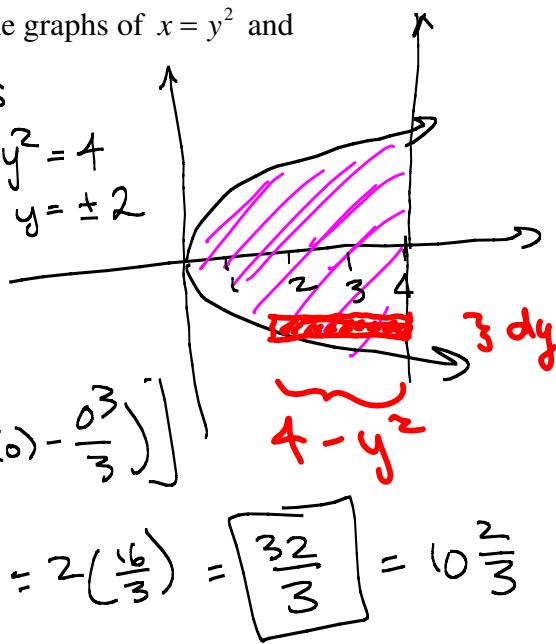
$$= 2 \left[4y - \frac{y^3}{3} \right] \Big|_0^2 = 2 \left[\left(4(2) - \frac{2^3}{3} \right) - \left(4(0) - \frac{0^3}{3} \right) \right]$$

$$= 2 \left[8 - \frac{8}{3} - 0 \right] = 2 \left[\frac{24}{3} - \frac{8}{3} \right] = 2 \left(\frac{16}{3} \right) = \boxed{\frac{32}{3}} = 10\frac{2}{3}$$

Find intersections

$$\text{Set } x's \text{ equal: } y^2 = 4$$

$$y = \pm 2$$

**Example 6:** Find the area of the region completely enclosed by the graphs of $x = 3 - y^2$ and $x = y + 1$.

$$x = y + 1$$

$$x - 1 = y$$

$$y = x - 1$$

$$\text{Area} = \int_{-2}^1 (3 - y^2 - (y + 1)) dy$$

$$= \int_{-2}^1 (-y^2 - y + 2) dy$$

$$= -\frac{y^3}{3} \Big|_{-2}^1 - \frac{y^2}{2} \Big|_{-2}^1 + 2y \Big|_{-2}^1$$

$$= -\frac{1}{3} + \frac{(-2)^3}{3} - \frac{1}{2} + \frac{(-2)^2}{2} + 2(1) - 2(-2)$$

$$= -\frac{1}{3} - \frac{8}{3} - \frac{1}{2} + \frac{4}{2} + 2 + 4 = -\frac{9}{3} + \frac{3}{2} + 6 = -3 + \frac{3}{2} + 6$$

$$= 3 + \frac{3}{2} = \frac{6}{2} + \frac{3}{2} = \boxed{\frac{9}{2}}$$

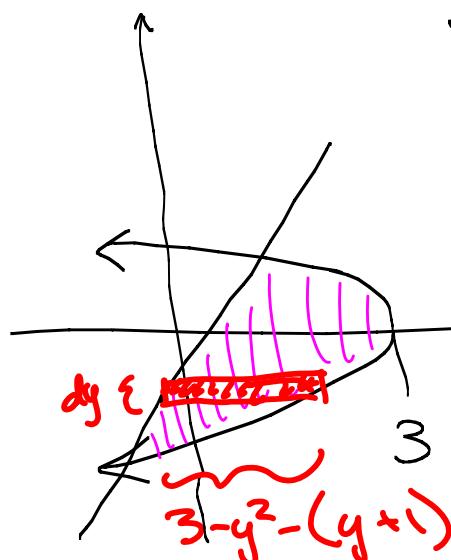
Find intersections:

$$3 - y^2 = y + 1$$

$$0 = y^2 + y - 2$$

$$0 = (y+2)(y-1)$$

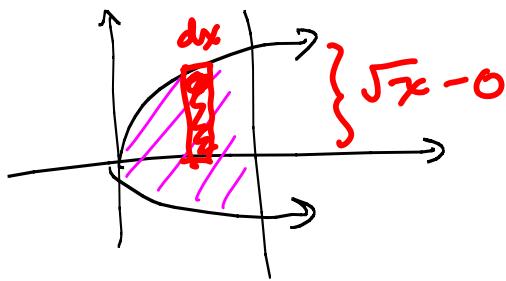
$$y = 1, -2$$



Ex 5:

Another way:

$$\text{Area} = 2 \int_0^4 \sqrt{x} dx = 2 \int_0^4 x^{1/2} dx$$



$$= 2 \left(\frac{x^{3/2}}{3/2} \right) \Big|_0^4 = \frac{4}{3} x^{3/2} \Big|_0^4 = \frac{4}{3} ((4)^{3/2} - (0)^{3/2})$$

Note: $\sqrt[3]{x^2} = (\sqrt[4]{x})^3 = (\sqrt[3]{x^3})^{1/2}$

$$= \frac{4}{3} (\sqrt[3]{1})^3 = \frac{4}{3} \cdot 2^3 = \frac{4}{3} (8)$$

$$= \boxed{\frac{32}{3}}$$