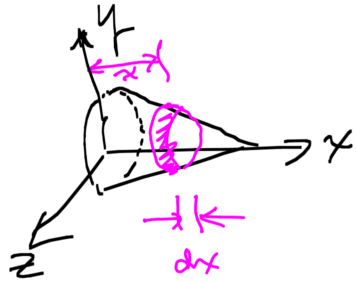
7.2: Volumes: The Disk Method
(Finding volumes of solids with known cross-sections--disks, washers, or other parallel slices)

Slicing by parallel planes (for solids with known cross sections):

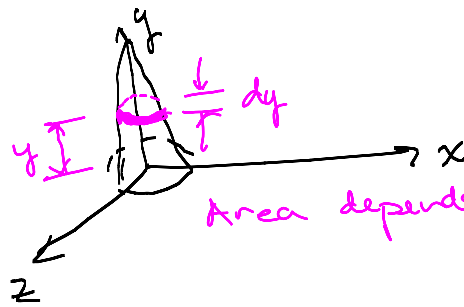
Cross sections of area $A(x)$ perpendicular to x-axis:

$$V = \int_a^b A(x)\,dx$$



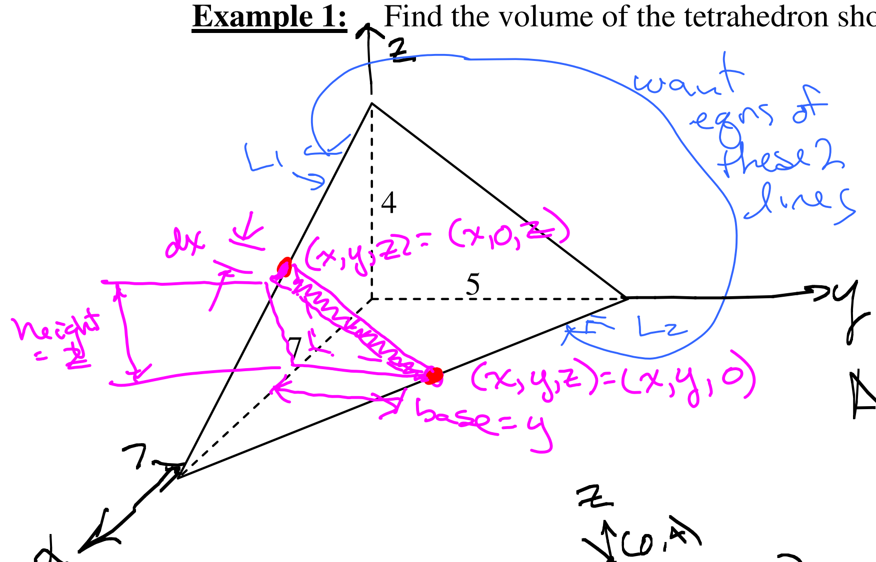
Cross sections of area $A(y)$ perpendicular to ~~x~~ y-axis:

$$V = \int_a^b A(y)\,dy$$



Area depends on y

Example 1: Find the volume of the tetrahedron shown.

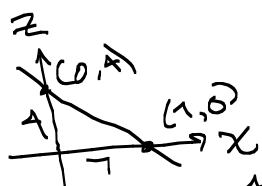


Need to write the area of a triangular slice as a function of x.

$$A_{\text{triangular slice}} = \frac{1}{2}(\text{base})(\text{height})$$
$$= \frac{1}{2}(y)(z)$$

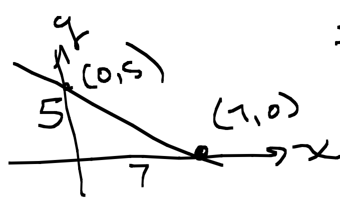
We want to get y and z in terms of x

Find eqn of Line L1:



$$\text{slope} = \frac{4-0}{0-7} = -\frac{4}{7}$$
$$z = mx + b$$
$$z = -\frac{4}{7}x + 4 \quad (\text{Eqn of } L1)$$

Find eqn of Line L_2:



$$\text{slope} = \frac{5-0}{0-7} = -\frac{5}{7}$$
$$y = mx + b$$
$$y = -\frac{5}{7}x + 5$$

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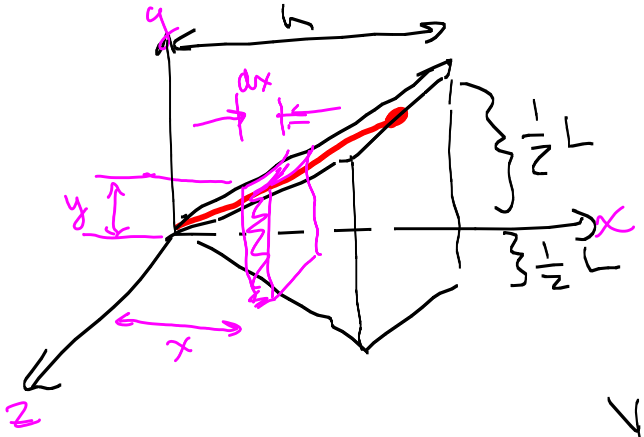
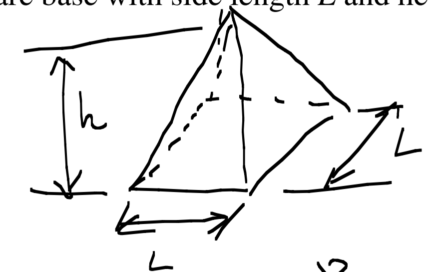
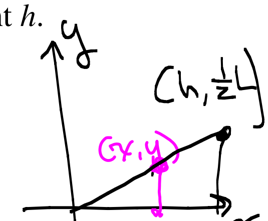
Example 2: Find the volume of a pyramid with square base with side length L and height h.







$$\text{slope} = \frac{\frac{1}{2}L}{h} = \frac{L}{2h}$$

$$V_{slice} = (\text{side length})^2 \Delta x = (2y)^2(\Delta x)$$

$$V = \int A_{slice}\,dx = \int (2y)^2\,dx$$

$$\text{Line: } y = \frac{L}{2h}x \quad (y = mx + b)$$

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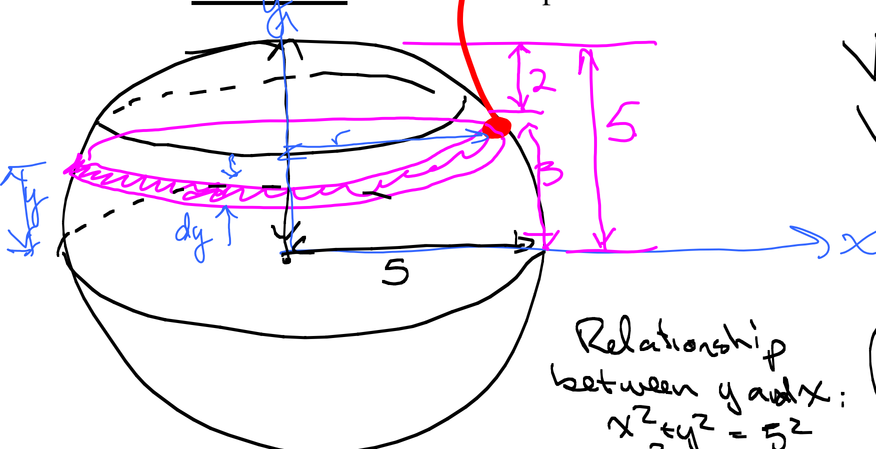
(x,y)

Example 3: Consider a sphere of radius 5. Find the volume of a cap of height 2.



Volume of 1 slice: $\pi r^2(\text{thickness})$

$V = \int \pi r^2\,dy$, where r = radius of 1 slice

$r = x$

$$= \int \pi x^2\,dy$$

Relationship between y and x:

$$x^2 + y^2 = 5^2$$
$$x^2 = 25 - y^2$$

$$= \int_3^5 \pi(25 - y^2)\,dy$$

$$= \pi \int_3^5 (25 - y^2)\,dy$$

~~**Example 4:** Consider a sphere of radius r. Find the volume of a cap of height h.~~

$$= \pi\left(25y - \frac{y^3}{3}\right)\Big|_3^5$$

$$= \pi\left(25(5) - \frac{5^3}{3} - \left(25(3) - \frac{3^3}{3}\right)\right)$$

$$= \pi\left(125 - \frac{125}{3} - 75 + 9\right)$$

$$= \pi\left(59 - \frac{125}{3}\right) = \pi\left(\frac{177}{3} - \frac{125}{3}\right)$$

$$= \pi\left(\frac{52}{3}\right) = \boxed{\frac{52\pi}{3}}$$

Ex 4: Find volume of whole sphere.

$$V = \pi\left(25y - \frac{y^3}{3}\right)\Big|_0^5 = \text{top half of sphere}$$

$$V_{sphere} = 2\pi\left(25y - \frac{y^3}{3}\right)\Big|_0^5$$

$$= 2\pi\left(25(5) - \frac{5^3}{3} - \left(25(0) - \frac{0^3}{3}\right)\right)$$

$$= 2\pi\left(\cancel{50}^{\,125} - \frac{125}{3} - 0\right)$$

$$= 2\pi\left(\frac{375}{3} - \frac{125}{3}\right) =$$

$$= 2\pi\left(\frac{250}{3}\right) = \boxed{\frac{500\pi}{3}}$$

From memorized formula,

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(5)^3$$

$$\frac{4\pi(125)}{3} = \boxed{\frac{500\pi}{3}}$$

The disk and washer method:

$$V \approx \sum_{i=1}^{n} A_i \Delta x \text{ and so } V = \lim_{n\to\infty} \sum_{i=1}^{n} A_i \Delta x .$$

Circular slice:
Area $= \pi R^2$

For circular cross-sectional areas:

$$V \approx \sum_{i=1}^{n} \pi R(x_i)^2 \Delta x \text{ and so } V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi R(x_i)^2 \Delta x .$$

Solids formed by rotation:

Horizontal Axis of Revolution:

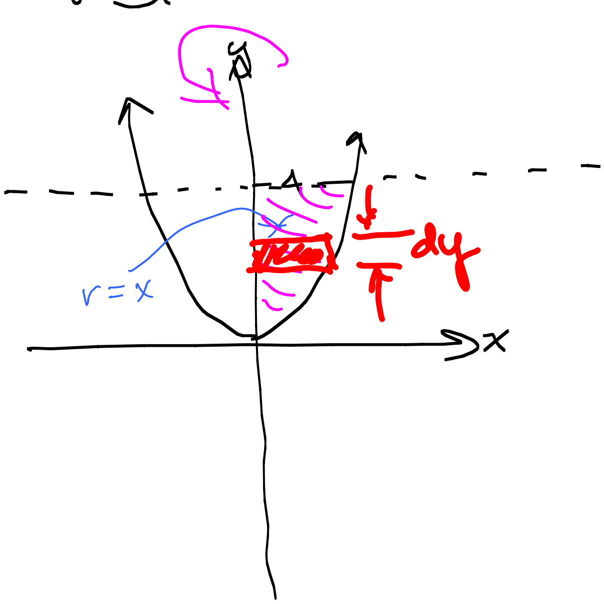
$$V = \pi \int_a^b [R(x)]^2 dx$$

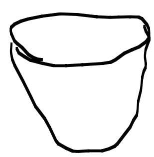
Vertical Axis of Revolution:

$$V = \pi \int_c^d [R(y)]^2 dy$$

y-axis

Example 5: Find the volume of the solid generated by revolving the area bounded by $y = x^2$, $x = 0$, and $y = 4$ around the y-axis.





Volume of 1 slice $= \pi r^2 \, dy$

$$V = \int \pi r^2 \, dy$$

$$= \int \pi x^2 \, dy$$

$y = x^2 \longrightarrow$

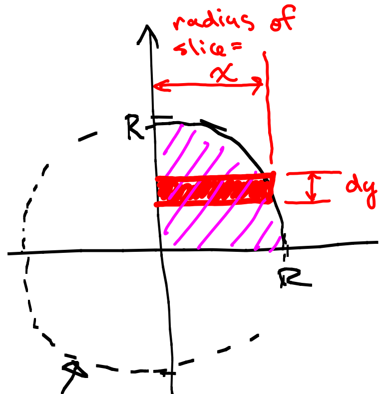
$$= \int_0^4 \pi (y) \, dy$$

$$= \pi \int_0^4 y \, dy = \frac{\pi y^2}{2} \Big|_0^4$$

$$= \frac{\pi (4)^2}{2} - \frac{\pi (0)^2}{2}$$

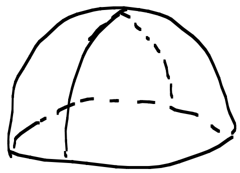
$$= \frac{16\pi}{2} = \boxed{8\pi}$$

Example 6: Find the volume of a sphere of radius R.



circle $x^2+y^2=R^2$

$x^2=R^2-y^2$



Top half of sphere

$$V_{slice} = \pi(\text{radius of slice})^2\,dy$$

$$V_{\substack{\text{top half}\\\text{of sphere}}} = \int_0^R \pi x^2\,dy$$

$$V_{\substack{\text{whole}\\\text{sphere}}} = 2\int_0^R \pi x^2\,dy$$

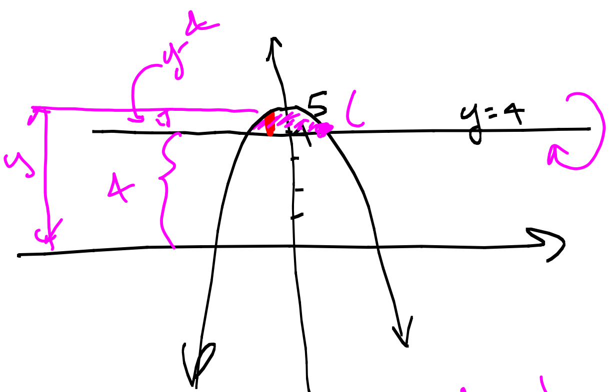
$$= 2\int_0^R \pi(R^2-y^2)\,dy$$

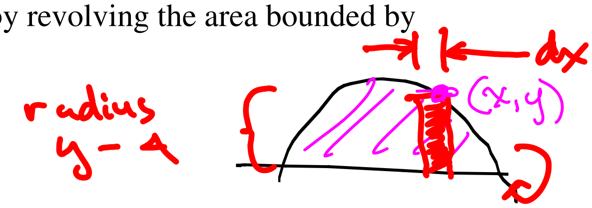
$$= 2\pi\int_0^R (R^2-y^2)\,dy = 2\pi\left[R^2y-\frac{y^3}{3}\right]\Big|_0^R$$

$$= 2\pi\left[R^2(R)-\frac{R^3}{3}-0\right] = 2\pi\left[R^3-\frac{R^3}{3}\right]$$

$$= 2\pi\left[\frac{2R^3}{3}\right] = \frac{4\pi R^3}{3} = \boxed{\frac{4}{3}\pi R^3}$$

Example 7: Find the volume of the solid generated by revolving the area bounded by $y=5-x^2$ and $y=4$ around the line $y=4$.





$$V_{slice} = \pi r^2\,dx$$

$$V_{total} = \int \pi r^2\,dx$$

$$= \int_{-1}^{1} \pi(y-4)^2\,dx$$

$y=5-x^2$

$$= \int_{-1}^{1} \pi(5-x^2-4)^2\,dx$$

$$= \pi\int_{-1}^{1}(1-x^2)^2\,dx$$

$$= \pi\int_{-1}^{1}(1-2x^2+x^4)\,dx$$

$$= 2\pi\int_0^1(1-2x^2+x^4)\,dx$$

Find intersection:

$5-x^2=4$

$5-4=x^2$

$1=x^2$

$x=\pm 1$

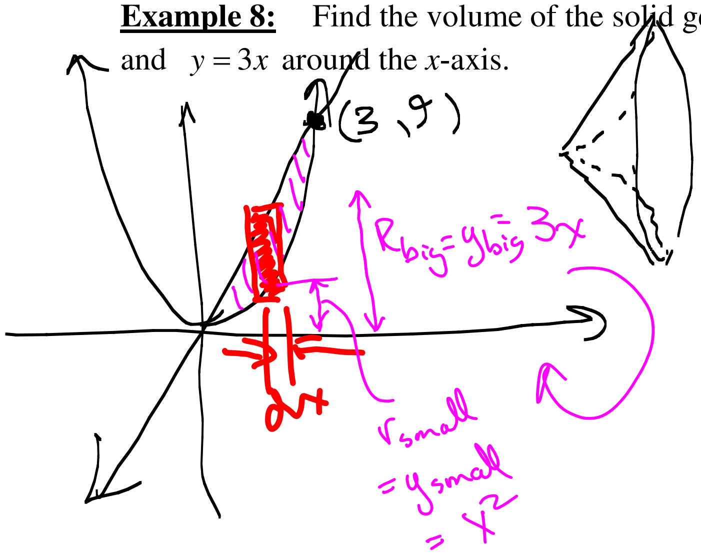
$$2\pi\left(x-\frac{2x^3}{3}+\frac{x^5}{5}\right)\Big|_0^1$$

$$2\pi\left(1-\frac{2(1)^3}{3}+\frac{1^5}{5}-0\right)$$

$$2\pi\left(1-\frac{2}{3}+\frac{1}{5}\right)$$

$$= 2\pi\left(\frac{15}{15}-\frac{10}{15}+\frac{3}{15}\right) = 2\pi\left(\frac{8}{15}\right) = \boxed{\frac{16\pi}{15}}$$

Example 8: Find the volume of the solid generated by revolving the area bounded by $y = x^2$ and $y = 3x$ around the x-axis.



$$V_{1\,slice} = \pi R_{big}^2 \Delta x - \pi r_{small}^2 \Delta x$$

$$V = \int \pi R_{big}^2\, dx - \int \pi r_{small}^2\, dx$$

$$= \pi \int_0^3 (3x)^2\, dx - \pi \int_0^3 (x^2)^2\, dx$$

Find intersection: Set y's equal

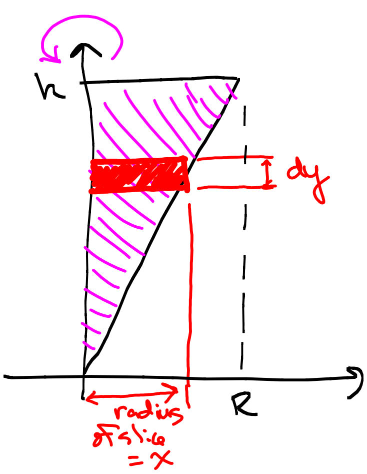
$x^2 = 3x$

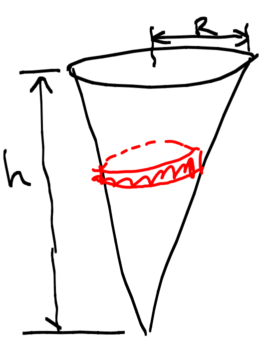
$x^2 - 3x = 0$

$x(x-3) = 0$

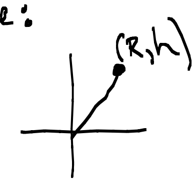
$x = 0,\ x = 3$

Example 9: Find the volume of a cone with height h and radius R.





Find eqn. of line:



slope $= \frac{h}{R}$

$y = mx + b$

$y = \frac{h}{R}x$

So $\frac{R}{h}y = x$

$$V_{slice} = \pi(\text{radius of slice})^2\, dy$$

$$V = \int \pi x^2\, dy$$

$$x = \frac{R}{h}y \longrightarrow \quad = \int_0^h \pi \left(\frac{R}{h}y\right)^2 dy$$

$$= \pi \frac{R^2}{h^2} \int_0^h y^2\, dy = \frac{\pi R^2}{h^2} \cdot \frac{y^3}{3} \Big|_0^h$$

$$= \frac{\pi R^2}{h^2}\left(\frac{h^3}{3} - \frac{0^3}{3}\right) = \frac{\pi R^2}{h^2} \cdot \frac{h^3}{3} = \frac{\pi R^2 h}{3} = \boxed{\frac{1}{3}\pi R^2 h}$$

I put the pointy end down so that my line would go through the origin.

Note: Compare this with Example 4 of Section 7.3 (Same cone, but used Shell method)

Ex 1 cont'd

Area of 1 slice:

$$A = \frac{1}{2}(\text{base})(\text{height})$$

$$= \frac{1}{2}(y)(z)$$

$$A(x) = \frac{1}{2}\left(-\frac{5}{7}x+5\right)\left(-\frac{4}{7}x+4\right)$$

$$\text{Volume} = \sum(\text{Areas of slices})(dx) = \sum \frac{1}{2}\left(-\frac{5}{7}x+5\right)\left(-\frac{4}{7}x+4\right)dx$$

$$V = \int_0^7 \underbrace{\frac{1}{2}\left(-\frac{5}{7}x+5\right)\left(-\frac{4}{7}x+4\right)}_{\text{Area of 1 slice}}\underbrace{dx}_{\text{thickness of 1 slice}}$$

$$= \frac{1}{2}\int_0^7\left(\frac{20}{49}x^2 - \frac{20}{7}x - \frac{20}{7}x + 20\right)dx$$

$$= \frac{1}{2}\int_0^7\left(\frac{20}{49}x^2 - \frac{40}{7}x + 20\right)dx$$

$$= \frac{1}{2}(20)\int_0^7\left(\frac{1}{49}x^2 - \frac{2}{7}x + 1\right)dx$$

$$= 10\left[\frac{1}{49}\cdot\frac{x^3}{3} - \frac{2}{7}\cdot\frac{x^2}{2} + x\right]\Big|_0^7$$

$$= 10\left[\frac{x^3}{147} - \frac{x^2}{7} + x\right]_0^7$$

$$= 10\left[\frac{7^3}{147} - \frac{7^2}{7} + 7 - 0\right]$$

$$\begin{array}{r} \overset{6}{4}9 \\ \times\ 7 \\ \hline 343 \end{array}$$

$$= 10\left[\frac{343}{147} - 7 + 7\right] = \frac{3430}{147} \approx \boxed{23.\overline{3}}\ \text{units}^3$$

Ex 2 cont'd: Line $y = \frac{h}{2L} x$

$$\text{Total Volume} = \int (2y)^2 \, dx = \int 4y^2 \, dx$$

$$= \int_0^h 4\left(\frac{L}{2h} x\right)^2 dx$$

$$= \int_0^h 4\left(\frac{L^2}{4h^2}\right) x^2 \, dx$$

$$= \frac{4L^2}{4h^2} \int_0^h x^2 \, dx$$

$$= \frac{L^2}{h^2} \int_0^h x^2 \, dx =$$

$$= \frac{L^2}{h^2} \cdot \frac{x^3}{3}\Big|_0^h = \frac{L^2}{h^2}\left(\frac{h^3}{3} - \frac{0^3}{3}\right)$$

$$= \frac{L^2 h^3}{3h^2} = \frac{L^2 h}{3}$$

$$= \boxed{\frac{1}{3} L^2 h}$$