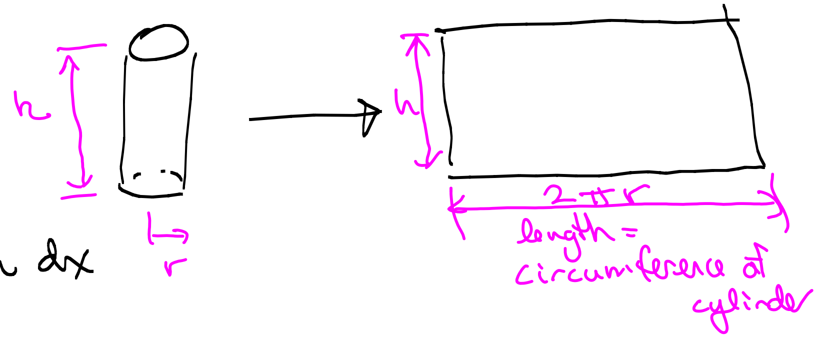
7.3: Volume: The Shell Method

Unroll a cylindrical shell



Finding volume by cylindrical shells:

Vertical Axis of Revolution:

$$V=\int_a^b 2\pi x\, f(x)\,dx \text{ where } a\le x\le b.$$

$\int 2\pi r h\, dx$

rectangle

Thickness of rectangle is either dx or dy

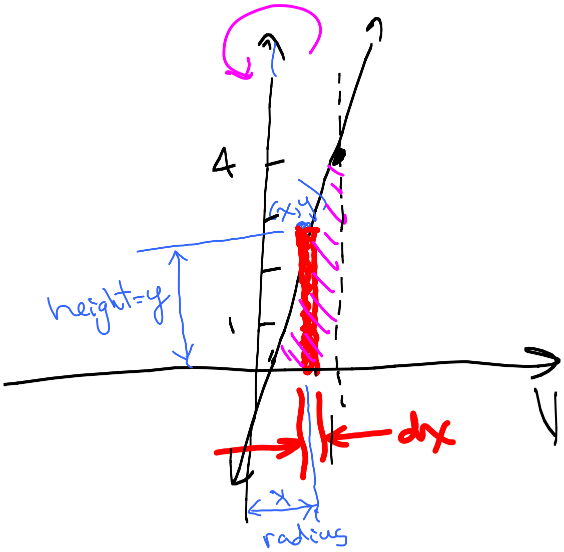
Horizontal Axis of Revolution:

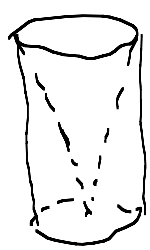
$$V=\int_c^d 2\pi y\, g(y)\,dy, \text{ where } c\le y\le d.$$

$\int 2\pi r h\, dy$

$\int 2\pi r h(\text{thickness})$

Example 1: Find the volume of the solid formed by rotating the region bounded by $y=4x$, $y=0$, and $x=1$ around the y-axis.



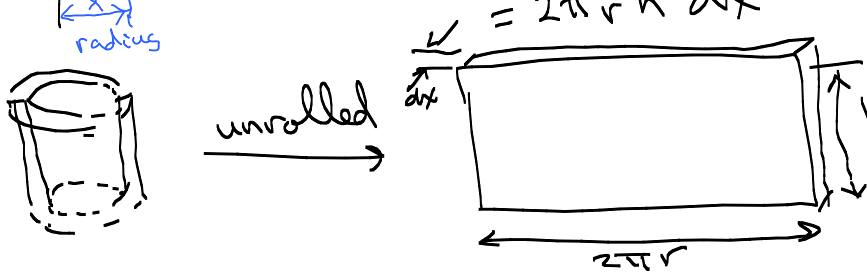


Slices: Representative rectangle is perpendicular to axis I'm rotating around

Shells: Representative Rectangle is parallel to axis I'm rotating around

$V_{1\,\text{shell}} = 2\pi r h(\text{thickness})$

$= 2\pi r h\, dx$



$V=\int 2\pi r h\, dx$

$V=\int_0^1 2\pi(x)(y)\, dx$

this dx means our integral must be in terms of x; and our limits of integration are x-values

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$V=\int_0^1 2\pi x(4x)\, dx$

$$V = 8\pi \int_0^1 x^2\,dx = 8\pi \cdot \frac{x^3}{3}\Big|_0^1 = 8\pi\left(\frac{1^3}{3} - \frac{0^3}{3}\right) = \boxed{\frac{8\pi}{3}}$$

Example 2: Find the volume of the solid formed by rotating the region bounded by $y = x$ and $x = \sqrt{y}$ around the x-axis.

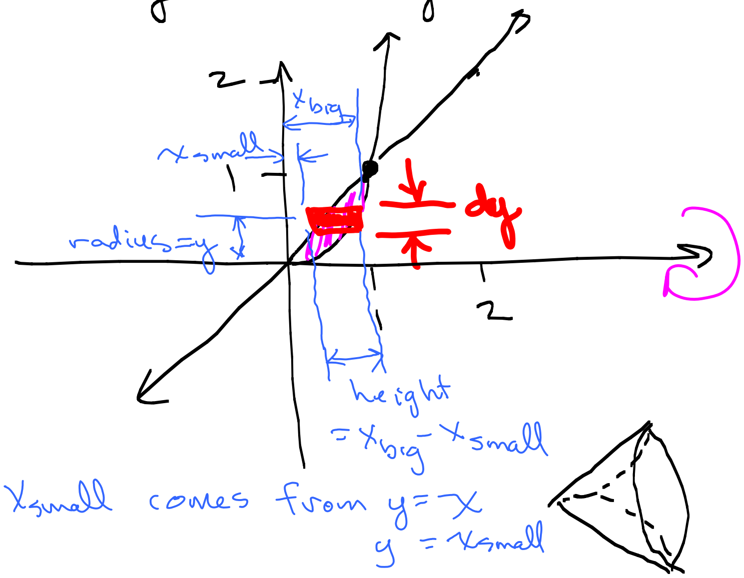
$x = \sqrt{y} \Rightarrow x^2 = y$

$x = \sqrt{y}$ is the right half of the parabola
$x = -\sqrt{y}$ is the left half

Find intersection pts:

$$\left.\begin{array}{l} y = x^2 \\ y = x \end{array}\right\} \to x^2 = x$$

$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0,\ x = 1$$



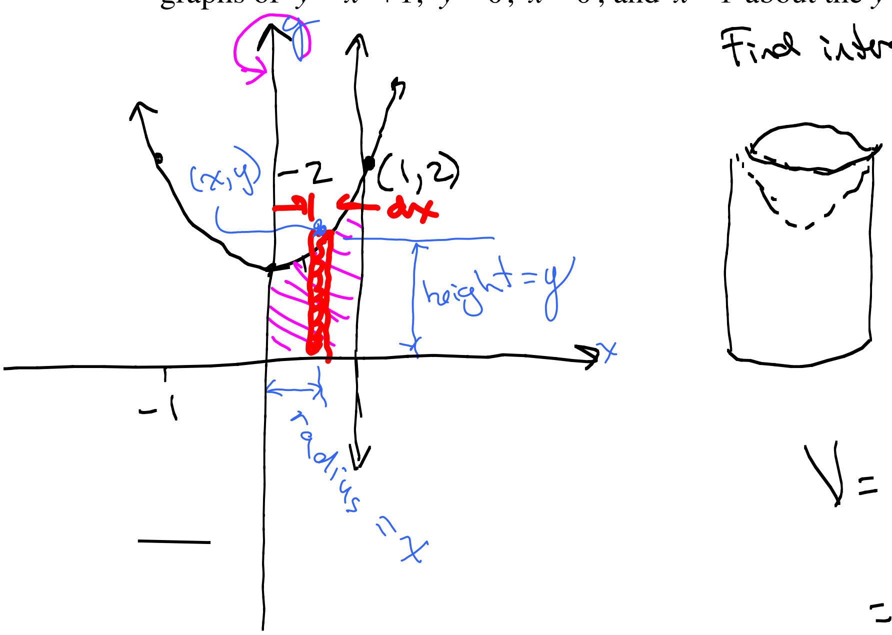
x_{small} comes from $y = x$
$y = x_{small}$

x_{big} comes from $x = \sqrt{y}$
$y = x^2$
$x_{big} = \sqrt{y}$

$$V_{shell} = 2\pi r h\,dy$$
$$= 2\pi(y)(x_{big} - x_{small})\,dy$$
$$= 2\pi(y)(\sqrt{y} - y)\,dy$$

$$V = \int_0^1 2\pi y(\sqrt{y} - y)\,dy$$
$$= 2\pi \int_0^1 (y^{3/2} - y^2)\,dy = 2\pi\left[\frac{y^{5/2}}{5/2} - \frac{y^3}{3}\right]\Big|_0^1$$

Example 3: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.



Find intersection of $y = x^2 + 1$ and $x = 1$
substitute $x = 1 \Rightarrow y = 1^2 + 1$
$y = 2$

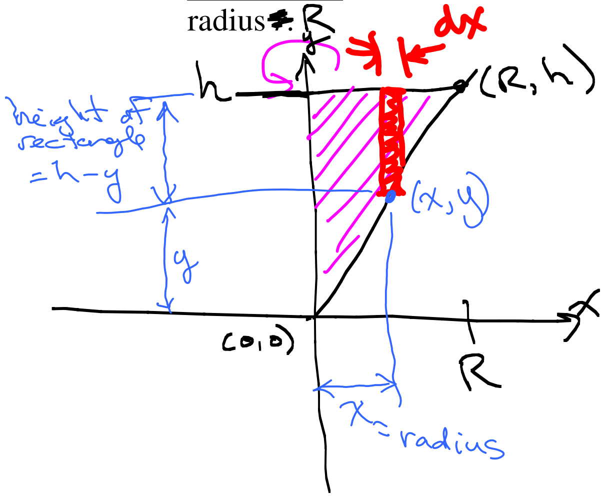
$$V_{shell} = 2\pi r h\,dx$$
$$= 2\pi(x)(y)\,dx$$

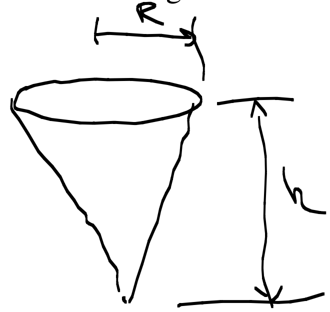
$$V = \int 2\pi(x)(y)\,dx$$
$$= \int_0^1 2\pi x(x^2+1)\,dx$$

$$= 2\pi \int_0^1 (x^3 + x)\,dx = 2\pi\left[\frac{x^4}{4} + \frac{x^2}{2}\right]\Big|_0^1 = 2\pi\left[\frac{1^4}{4} + \frac{1^2}{2} - 0\right]$$

$$= 2\pi\left[\frac{1}{4} + \frac{1}{2}\right] = 2\pi\left(\frac{3}{4}\right) = \frac{6\pi}{4} = \boxed{\frac{3\pi}{2}}$$

Example 4: Use the shell method to find the volume of a right circular cone of height h and radius R.





Find eqn of line

$$\text{slope} = \frac{h-0}{R-0} = \frac{h}{R}$$

y-intercept: 0

Eqn of line:

$$y = \frac{h}{R}x \qquad (y = mx + b)$$

$$V_{shell} = 2\pi \left(\begin{smallmatrix}\text{radius}\\ \text{of shell}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{height}\\ \text{of shell}\end{smallmatrix}\right) dx$$

$$= 2\pi\,(x)\,(h-y)\,dx$$

$$V = \int_0^R 2\pi x(h-y)\,dx = \int_0^R 2\pi x\left(h - \frac{h}{R}x\right)dx = 2\pi\int_0^R \left(hx - \frac{h}{R}x^2\right)dx$$

$$= 2\pi\left[h\frac{x^2}{2} - \frac{h}{R}\cdot\frac{x^3}{3}\right]_0^R = 2\pi\left[\frac{hR^2}{2} - \frac{hR^3}{3R}\right] = 2\pi\left[\frac{3hR^2}{6} - \frac{2hR^2}{6}\right]$$

$$= 2\pi\left(\frac{hR^2}{6}\right) = \frac{2\pi hR^2}{6} = \boxed{\frac{1}{3}\pi R^2 h}$$

Example 5: Find the volume of the solid formed by revolving the region bounded by the graphs of $y = 4x - x^2$, $y = 8x - 2x^2$ about the line $x = -2$.

Complete the square to graph the parabolas:

$y = 4x - x^2$

$y = -x^2 + 4x$

$y = -(x^2 - 4x)$

$y = -(x^2 - 4x + 4) + 4$

$\left(\frac{-4}{2}\right)^2 = 4$

$y = -(x-2)^2 + 4$

Vertex: $(2, 4)$

opens down

Find x-intercepts:

$0 = 4x - x^2$

$0 = -x^2 + 4x$

$0 = -x(x-4)$

$x = 0, 4$

$y = 8x - 2x^2$

$y = -2x^2 + 8x$

$y = -2(x^2 - 4x)$

$y = -2(x^2 - 4x + 4) + 8$

$\left(\frac{-4}{2}\right)^2$

$y = -2(x-2)^2 + 8$

Vertex: $(2, 8)$

opens down

Find x-intercepts

$0 = -2x^2 + 8x$

$0 = -2x(x-4)$

$x = 0, 4$

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Ex 2 cont'd:

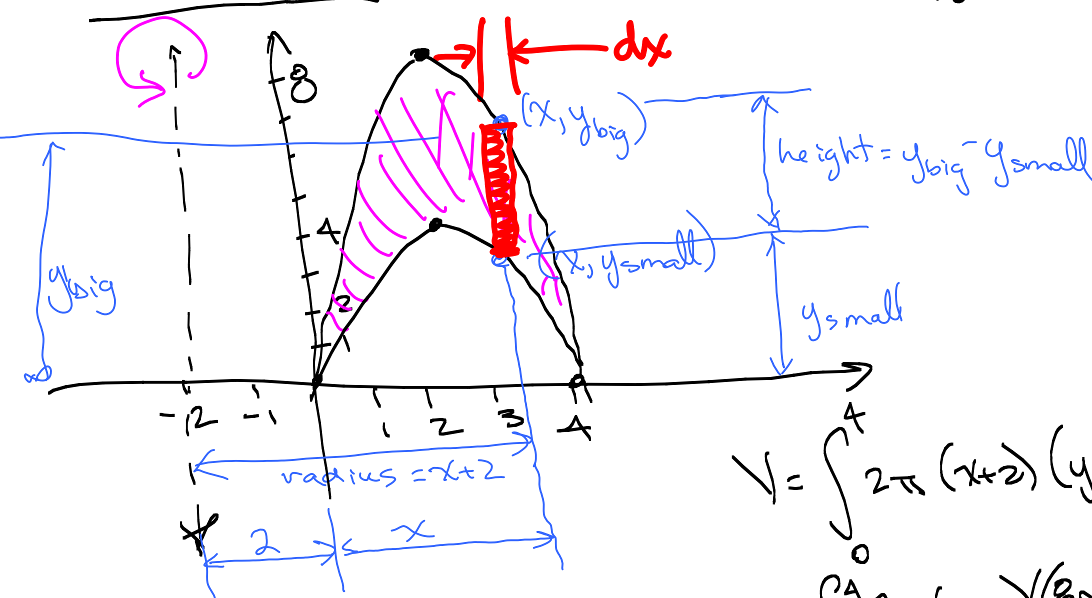
$$V = 2\pi\left[\frac{2}{5}y^{5/2} - \frac{y^3}{3}\right]_0^1 = 2\pi\left(\frac{2(1)^{5/2}}{5} - \frac{1^3}{3} - 0\right)$$

$$= 2\pi\left(\frac{2}{5} - \frac{1}{3}\right) = 2\pi\left(\frac{6}{15} - \frac{5}{15}\right)$$

$$= 2\pi\left(\frac{1}{15}\right)$$

$$= \boxed{\frac{2\pi}{15}}$$

Ex 5 cont'd:



$$V_{shell} = 2\pi r h\, dx$$

$$= 2\pi(x+2)(y_{big} - y_{small})\, dx$$

$$V = \int_0^4 2\pi(x+2)(y_{big} - y_{small})\, dx$$

$$= \int_0^4 2\pi(x+2)\big((8x - 2x^2) - (4x - x^2)\big)\, dx$$

$$= 2\pi\int_0^4 (x+2)(4x - x^2)\, dx = 2\pi\int_0^4 (4x^2 - x^3 + 8x - 2x^2)\, dx$$

$$= 2\pi\int_0^4 (2x^2 - x^3 + 8x)\, dx = 2\pi\left[\frac{2x^3}{3} - \frac{x^4}{4} + \frac{8x^2}{2}\right]_0^4$$

$$= 2\pi\left[\frac{2(4)^3}{3} - \frac{4^4}{4} + \frac{\cancel{8}^{4}(4^2)}{\cancel{2}_{1}} - 0\right] = 2\pi\left[\frac{128}{3} - 64 + 64\right]$$

$$= 2\pi\left[\frac{128}{3}\right] = \boxed{\frac{256\pi}{3}}$$