

### 7.3: Volume: The Shell Method

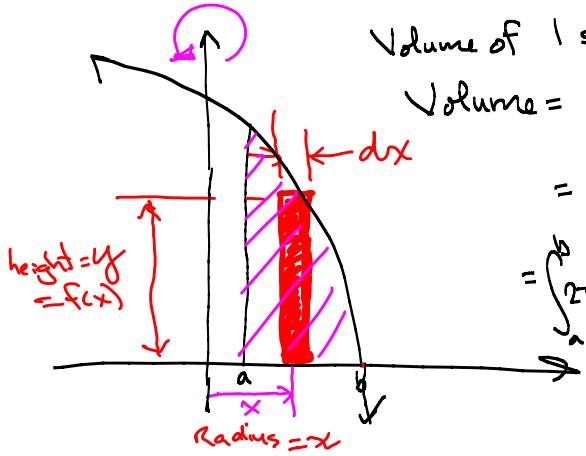
Finding volume by cylindrical shells:

Vertical Axis of Revolution:

$$V = \int_a^b 2\pi x f(x) dx \text{ where } a \leq x \leq b.$$

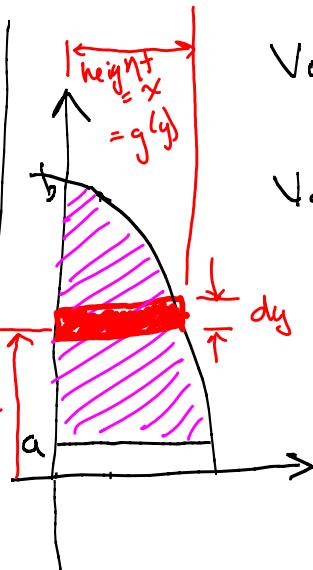
Horizontal Axis of Revolution:

$$V = \int_c^d 2\pi y g(y) dy, \text{ where } c \leq y \leq d.$$



$$\text{Volume of 1 shell} = 2\pi r h \Delta x$$

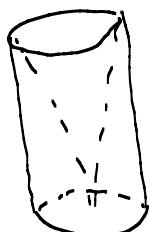
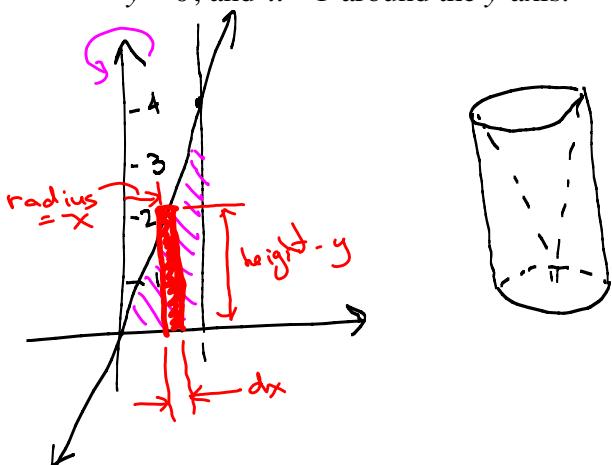
$$\begin{aligned}\text{Volume} &= \int 2\pi r h dx \\ &= \int_a^b 2\pi x y dx \\ &= \int_a^b 2\pi x f(x) dx\end{aligned}$$



$$\begin{aligned}\text{Volume of 1 shell} &= 2\pi r h \Delta y \\ \text{Volume} &= \int_a^b 2\pi r h dy \\ &= \int_a^b 2\pi y \times dy \\ &= \int_a^b 2\pi y g(y) dy\end{aligned}$$

**Example 1:** Find the volume of the solid formed by rotating the region bounded by  $y = 4x$ ,  $y = 0$ , and  $x = 1$  around the  $y$ -axis.

$$V_{\text{shell}} = 2\pi r h dx$$

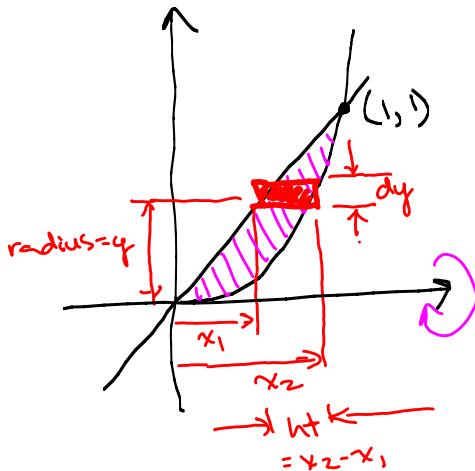


$$\begin{aligned}V &= \int 2\pi r h dx \\ &= \int_0^1 2\pi x y dx \\ &= \int_0^1 2\pi x (4x) dx = 8\pi \int_0^1 x^2 dx \\ &= 8\pi \left(\frac{x^3}{3}\right) \Big|_0^1 = 8\pi \left(\frac{1^3}{3} - \frac{0^3}{3}\right) \\ &= 8\pi \left(\frac{1}{3}\right) = \boxed{\frac{8\pi}{3}}\end{aligned}$$

$$x = \sqrt{y} \Rightarrow x^2 = y$$

$x = \sqrt{y}$  is right half 7.3.2

**Example 2:** Find the volume of the solid formed by rotating the region bounded by  $y = x$  and  $x = \sqrt{y}$  around the  $x$ -axis.



Find intersection:  $x = \sqrt{y}$

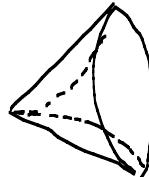
$$y = x \Rightarrow x = \sqrt{x}$$

Square both sides:  $x^2 = x$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0, x = 1$$



$$V_{\text{shell}} = 2\pi r h dy$$

$$V = \int 2\pi r h dy$$

$$= \int 2\pi y (x_2 - x_1) dy$$

$$= \int_0^1 2\pi y (\sqrt{y} - y) dy = 2\pi \int_0^1 (y^{3/2} - y^2) dy$$

$$= 2\pi \left( \frac{y^{5/2}}{5} - \frac{y^3}{3} \right) \Big|_0^1 = 2\pi \left( \frac{2}{5} - \frac{1}{3} \right)$$

$$= 2\pi \left( \frac{2}{5} - \frac{1}{3} - 0 \right) = 2\pi \left( \frac{2}{5} - \frac{1}{3} \right) = 2\pi \left( \frac{6}{15} - \frac{5}{15} \right)$$

**Example 3:** Find the volume of the solid formed by revolving the region bounded by the graphs of  $y = x^2 + 1$ ,  $y = 0$ ,  $x = 0$ , and  $x = 1$  about the  $y$ -axis.

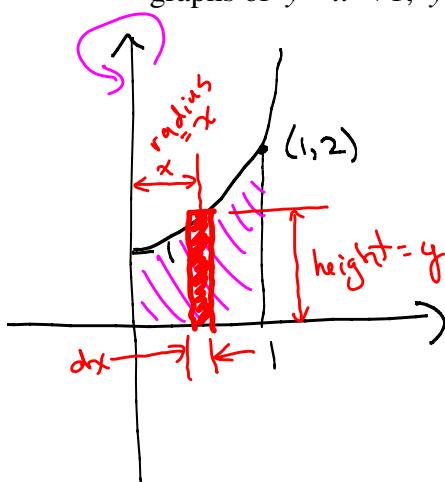
Find intersection:  $y = x^2 + 1$

$$x = 1 \Rightarrow y = 1^2 + 1$$

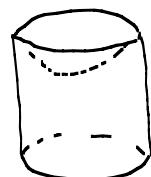
$$y = 2$$

$$= 2\pi \left( \frac{1}{5} \right)$$

$$= \frac{2\pi}{15}$$



(cylinder with bowl-shaped depression in the top)



$$V_{\text{shell}} = 2\pi r h dx$$

$$V = \int 2\pi r h dx$$

$$= \int_0^1 2\pi x y dx$$

$$= \int_0^1 2\pi x (x^2 + 1) dx$$

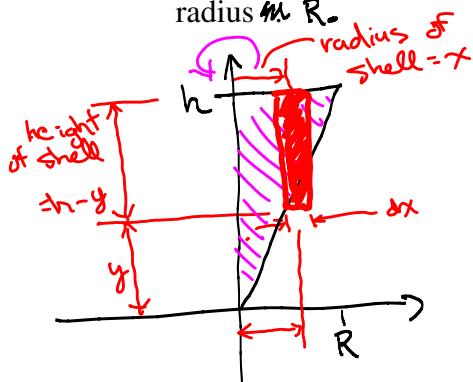
$$= 2\pi \int_0^1 (x^3 + x) dx = 2\pi \left( \frac{x^4}{4} + \frac{x^2}{2} \right) \Big|_0^1$$

$$= 2\pi \left( \frac{1}{4} + \frac{1}{2} - 0 \right) = 2\pi \left( \frac{3}{4} \right) = \frac{6\pi}{4}$$

$$= \boxed{\frac{3\pi}{2}}$$

Note: This one is much easier with shells than with disks/washers!

**Example 4:** Use the shell method to find the volume of a right circular cone of height  $h$  and radius  $R$ .



I chose to put the pointy end down so that my line would go through the origin.



Find eqn of line: slope =  $\frac{h}{R}$

$$y = mx + b$$

$$y = \frac{h}{R}x + 0$$

$$y = \frac{h}{R}x$$

$$V_{\text{shell}} = 2\pi rh dx$$

$$V = \int 2\pi rh dx = \int_0^R 2\pi x (h-y) dx$$

$$= \int_0^R 2\pi x \left(h - \frac{h}{R}x\right) dx = 2\pi \int_0^R \left(hx - \frac{h}{R}x^2\right) dx$$

$$= 2\pi \left[ \frac{hx^2}{2} - \frac{h}{R} \cdot \frac{x^3}{3} \right]_0^R = 2\pi \left[ \frac{hx^2}{2} - \frac{hx^3}{3R} \right]_0^R$$

$$= 2\pi \left[ \frac{hR^2}{2} - \frac{hR^3}{3R} - 0 \right] = 2\pi \left[ \frac{hR^2}{2} - \frac{hR^2}{3} \right] = 2\pi \left[ \frac{3hR^2}{6} - \frac{2hR^2}{6} \right]$$

**Example 5:** Find the volume of the solid formed by revolving the region bounded by the graphs of  $y = 4x - x^2$ ,  $y = 8x - 2x^2$  about the line  $x = -2$ .

Complete the square to find the vertices of the parabolas:

$$y = -x^2 + 4x$$

$$y = -(x^2 - 4x)$$

$$y = -(x^2 - 4x + 4) + 4$$

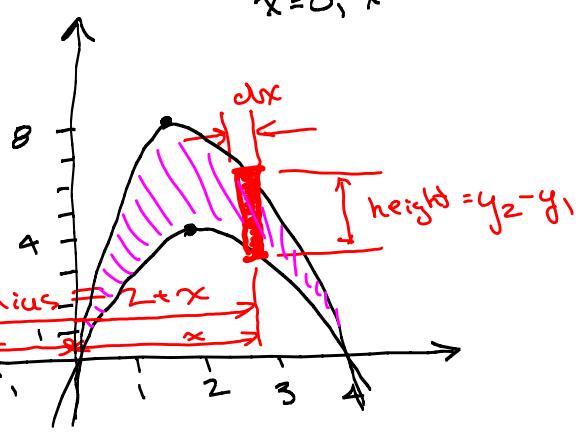
$$y = -(x-2)^2 + 4$$

vertex  $(2, 4)$   
opens down

$$\text{Find } x\text{-intercepts: } 0 = -x^2 + 4x$$

$$0 = -x(x-4)$$

$$x = 0, x = 4$$



$$y = -2x^2 + 8x$$

$$y = -2(x^2 - 4x)$$

$$y = -2(x^2 - 4x + 4) + 8$$

$$y = -2(x-2)^2 + 8$$

vertex  $(2, 8)$   
opens down

Find x-intercepts:

$$0 = -2x^2 + 8x$$

$$0 = -2x(x-4)$$

$$x = 0, x = 4$$



ugh, I cannot draw it... Have you seen a bundt pan?  
Think of a flat plate with a parabolic hump going around the outer half.

$$V_{\text{shell}} = 2\pi rh \, dx$$

$$\begin{aligned}
 V &= \int 2\pi rh \, dx \\
 &= \int 2\pi x (y_2 - y_1) \, dx = 2\pi \int_0^4 (2+x)(8x-2x^2) - (4x-x^2) \, dx \\
 &= 2\pi \int_0^4 (2+x)(8x-2x^2 - 4x + x^2) \, dx = 2\pi \int_0^4 (x+2)(-x^2 + 4x) \, dx \\
 &= 2\pi \int_0^4 (-x^3 + 4x^2 - 2x^2 + 8x) \, dx = 2\pi \int_0^4 (-x^3 + 2x^2 + 8x) \, dx \\
 &= 2\pi \left[ -\frac{x^4}{4} + \frac{2x^3}{3} + \frac{8x^2}{2} \right] \Big|_0^4 = 2\pi \left[ -\frac{4^4}{4} + 2 \frac{(4^3)}{3} + 4(4)^2 - 0 \right] \\
 &= 2\pi \left[ -64 + \frac{128}{3} + 64 \right] = 2\pi \left[ 0 + \frac{128}{3} \right] \\
 &= 2\pi \left[ \frac{128}{3} \right] = \boxed{\frac{256\pi}{3}}
 \end{aligned}$$

Note: This problem would be a serious pain using disks/washers. Shells are much easier!