

2.4: The Chain Rule

The chain rule allows us to find the derivative of many more functions.

Example 1: Suppose $g(x) = (x^3 + 2)^2$. Find $g'(x)$.

$$g(x) = (x^3 + 2)^2$$

The chain rule lets us differentiate composite functions.

The Chain Rule:

If f and g are both differentiable and $F(x) = f(g(x))$, then F is differentiable and

$$F'(x) = f'(g(x))g'(x).$$

Alternatively, if $y = f(u)$ and $u = f(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

With the chain rule, we take the derivative of the “outer function” and multiply by the derivative of the “inner function”.

Example 2: Find the derivative of $h(x) = (x^3 + 2)^{50}$.

Example 3: Find the derivative of $f(x) = \sqrt{2x^3 - 5x}$.

Example 4: Suppose that $y = \frac{2}{(3x^5 - 4)^3}$. Find $\frac{dy}{dx}$.

Example 5: Find the derivative of $f(x) = \sqrt[3]{\cos x}$.

Example 6: Find the derivative of $g(x) = \cos \sqrt[3]{x}$.

Example 7: Suppose that $y = \frac{1}{\cos x}$. Find $\frac{dy}{dx}$.

Example 8: Suppose that $h(x) = (3x^2 - 4)^3(2x - 9)^2$. Find $h'(x)$.

Example 9: Suppose that $f(x) = \left(\frac{2x+1}{2x-1}\right)^5$. Find $f'(x)$.

Example 10: Find the derivative of $y = \cos(\sin(\pi x^2))$.

Example 11: Find the derivative of $g(x) = \frac{\cos^2 x}{\sqrt{4x+1}}$.

Example 12: Find the first and second derivatives of $f(x) = (x^2 + 4)^5$.

$$\begin{aligned} f'(x) &= 5(x^2 + 4)^4 \frac{d}{dx}(x^2 + 4) \\ &= 5(x^2 + 4)^4(2x) \\ &= 10x(x^2 + 4)^4 \end{aligned}$$

$\begin{aligned} f''(x) &= 10x \frac{d}{dx}(x^2 + 4)^4 + (x^2 + 4)^4 \frac{d}{dx}(10x) \\ &= 10x(4)(x^2 + 4)^3 \frac{d}{dx}(x^2 + 4) + (x^2 + 4)^4(10) \\ &= 40x(x^2 + 4)^3(2x) + 10(x^2 + 4)^4 \end{aligned}$

Example 13: Find the first and second derivatives of $f(x) = \cos(3x^2)$.

$$\begin{aligned} f'(x) &= -\sin(3x^2) \frac{d}{dx}(3x^2) \quad [\text{chain rule}] \\ &= [-\sin(3x^2)](6x) \\ &= [-6x \sin(3x^2)] \end{aligned}$$

Note:
 For $y = \cos x$
 $\frac{dy}{dx} = -\sin x$
 $y = (3x^2 + 5)^4$
 $\frac{dy}{dx} = 4(3x^2 + 5)^3 \frac{d}{dx}(3x^2 + 5)$
 $= 4(3x^2 + 5)^3(6x)$

$$\begin{aligned} f''(x) &= -6x \frac{d}{dx}[\sin(3x^2)] + [\sin(3x^2)] \frac{d}{dx}(-6x) \\ &= -6x[\cos(3x^2)](6x) + [\sin(3x^2)](-6) \\ &= [-36x^2 \cos(3x^2) - 6 \sin(3x^2)] \\ &= [-6(6x^2 \cos(3x^2) + \sin(3x^2))] \end{aligned}$$

Rewrite: Example 14: Find the first and second derivatives of $y = \frac{2}{(3x+5)^2}$

$$y = 2(3x+5)^{-2}$$

$$\begin{aligned} y' &= \frac{dy}{dx} = -4(3x+5)^{-3} \frac{d}{dx}(3x+5) \\ &= -4(3x+5)^{-3}(3) = \boxed{-12(3x+5)^{-3}} \end{aligned}$$

$$\begin{aligned} y'' &= \frac{d^2y}{dx^2} = 36(3x+5)^{-4}(3) \quad \text{or} \quad \boxed{-\frac{12}{(3x+5)^3}} \\ &= \boxed{108(3x+5)^{-4}} = \boxed{\frac{108}{(3x+5)^4}} \end{aligned}$$

Example 15: Find the equation of the tangent line to $y = \frac{x^2}{\sin \pi x + 1}$ at the point where $x=1$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(\sin \pi x + 1)\frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin \pi x + 1)}{(\sin \pi x + 1)^2} \\ &= \frac{(\sin \pi x + 1)(2x) - x^2([\cos(\pi x)](\pi) + 0)}{(\sin \pi x + 1)^2} \\ &= \frac{2x(\sin \pi x + 1) - \pi x^2 \cos(\pi x)}{(\sin \pi x + 1)^2} \\ \left. \frac{dy}{dx} \right|_{x=1} &= \frac{2(1)(\sin \pi + 1) - \pi(1)^2 \cos \pi}{(\sin \pi + 1)^2} = \frac{2(0+1) - \pi(-1)}{(0+1)^2} \\ &= \frac{2 + \pi}{1} = 2 + \pi \Rightarrow \text{slope} \end{aligned}$$

Need a point: Have $x=1$, must find y .

$$y = \frac{x^2}{\sin \pi x + 1}, \text{ so } y \Big|_{x=1} = \frac{1^2}{\sin \pi + 1} = \frac{1}{0+1} = 1$$

Point: $(1, 1)$ } $\left\{ \Rightarrow \begin{cases} y - y_1 = m(x - x_1) \\ y - 1 = (2 + \pi)(x - 1) \end{cases} \right.$

2.3 #95

$$f(x) = \frac{x}{x-1}$$

Find $f''(x)$

$$\begin{aligned} f'(x) &= \frac{(x-1)\frac{d}{dx}(x) - x\frac{d}{dx}(x-1)}{(x-1)^2} \\ &= \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{x-1-x}{(x-1)^2} \\ &= \frac{-1}{(x-1)^2} = -1(x-1)^{-2} \end{aligned}$$

$$f''(x) = 2(x-1)^3 \frac{d}{dx}(x-1)$$

$$= 2(x-1)^{-3}(1)$$

$$= \boxed{\frac{2}{(x-1)^3}}$$

same

$$\text{with quotient rule: } f''(x) = \frac{(x-1)^2 \frac{d}{dx}(-1) - (-1)\frac{d}{dx}(x-1)^2}{[(x-1)^2]^2}$$

$$= \frac{(x-1)^2(0) + 2(x-1)(-1)}{(x-1)^4} = \frac{2(x-1)}{(x-1)^4}$$

$$= \boxed{\frac{2}{(x-1)^3}}$$

Note: We would not use product rule
to differentiate $f(x) = 7x^3$

Ex: $g(x) = \frac{x^4}{5}$ & Don't use quotient rule:

Rewrite: $g(x) = \frac{1}{5}x^4$

$$g'(x) = \frac{1}{5}(4x^3) = \boxed{\frac{4}{5}x^3}$$

2.4 #8) (modified)

$f(x) = \sqrt{25-x^2}$. Find eqn of tangent line at $(-3, 4)$.

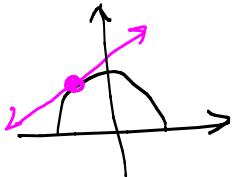
Find slope: $f(x) = (25-x^2)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(25-x^2)^{-\frac{1}{2}} \frac{d}{dx}(25-x^2)$$

$$= \frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{25-x^2}}$$

Find slope at $(-3, 4)$:

$$m = f'(-3) = -\frac{-3}{\sqrt{25-(-3)^2}} = \frac{3}{\sqrt{25-9}} = \frac{3}{4}$$



$$y = mx + b \quad \text{or} \quad y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{3}{4}(x - (-3))$$

$$y - 4 = \frac{3}{4}(x + 3)$$

$$y - 4 = \frac{3}{4}x + \frac{9}{4}$$

$$y = \frac{3}{4}x + \frac{9}{4} + 4$$

$$\boxed{y = \frac{3}{4}x + \frac{25}{4}}$$

Multiply by 4 to
clear fractions:

$$4y = 3x + 25$$

$$-3x + 4y - 25 = 0$$

(I think this is how harsoncalculus.com wrote the answer.)