

## 2.6: Related rates

General idea for solving rate problems:

1. Draw a sketch if applicable. The only dimensions you put on your sketch should be those that do not change.
2. Write down, in calculus notation, the rates you know and want.
3. Write an equation relating the quantities that are changing.
4. Differentiate it implicitly, with respect to time.
5. Substitute known quantities.
6. Solve for the required rate.

**Example 1:** The radius of a sphere is increasing at the rate of 2 inches per minute. Find the rate of change in volume when the radius is 6 inches.

Know:  $\frac{dr}{dt} = +2 \text{ in/min}$

Want:  $\frac{dV}{dt}$

When:  $r = 6 \text{ in}$

$r$  = radius

$V$  = volume

Write an equation relationship between the quantities in want and know,

$$V = \frac{4}{3}\pi r^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi (3r^2 \frac{dr}{dt})$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} \Big|_{r=6} = 4\pi (6 \text{ in})^2 \left(\frac{2 \text{ in}}{\text{min}}\right)$$

$$= 144\pi \cdot 2 \frac{\text{in}^3}{\text{min}}$$

$$= 288\pi \text{ in}^3/\text{min}$$

**Example 2:** A conical tank (with vertex down) has a diameter of 10 feet at the top and is 12 feet deep. If water is draining out at 10 cubic feet per minute, what is the rate of change in depth when the water is 8 feet deep?

Know:  $\frac{dV}{dt} = -10 \text{ ft}^3/\text{min}$

Want:  $\frac{dw}{dt}$

When:  $w = 8 \text{ ft}$

$V$  = volume of water

Need eqn relating  $V$  and  $w$  <sup>only</sup>

Volume of a cone

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

$r$  = radius  
 $h$  = height

$$\frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi r^2 w$$

Must get rid of  $r$ .  
get  $r$  in terms of  $w$

$$\frac{5}{12} = \frac{r}{w} \quad \text{or} \quad \frac{5}{r} = \frac{12}{w}$$

Solve for  $r$ :

$$\frac{5}{12}w = r \quad \text{or} \quad 12r = 5w$$

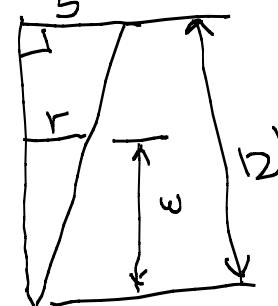
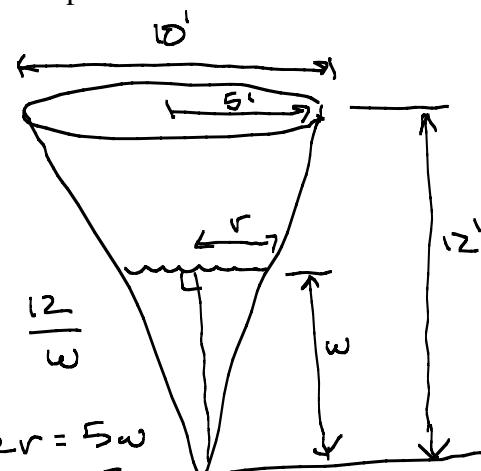
$$r = \frac{5}{12}w$$

$$V = \frac{1}{3}\pi r^2 w$$

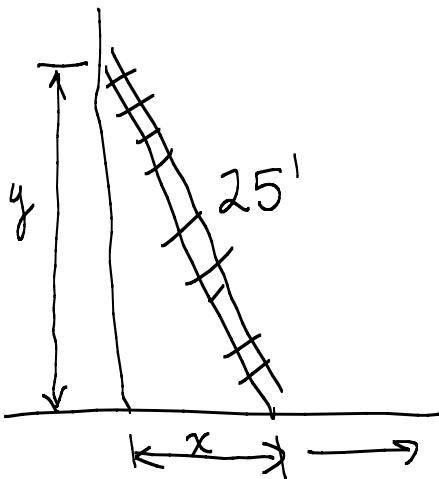
$$= \frac{1}{3}\pi \left(\frac{5}{12}w\right)^2 w$$

$$V = \frac{25\pi}{432} w^3$$

cont'd at end of notes



**Example 3:** A ladder 25 feet long is leaning against a wall. The base of the ladder is pulled away at 2 feet per second. How fast is the top of the ladder moving when the base is 9 feet away? What about when it is 24 feet away?



Need eqn relating  $x$  and  $y$ :

$$x^2 + y^2 = (25 \text{ ft})^2$$

$$x^2 + y^2 = 625 \text{ ft}^2$$

$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(625 \text{ ft}^2)$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$2y \frac{dy}{dt} = -2x \frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{-2x \frac{dx}{dt}}{2y} = -\frac{x \frac{dx}{dt}}{y}$$

Know:  $\frac{dx}{dt} = +2 \text{ ft/s}$

Want:  $\frac{dy}{dt}$

When:  $x = 9 \text{ ft}$   
 $x = 24 \text{ ft}$

cont'd at end of note

**Example 4:** A particle is moving along the parabola  $y^2 = 4x + 8$ . As it passes through the point  $(7, 6)$ , its  $y$ -coordinate is increasing at the rate of 3 units per second. How fast is the  $x$ -coordinate changing at this instant?

$$y^2 = 4x + 8$$

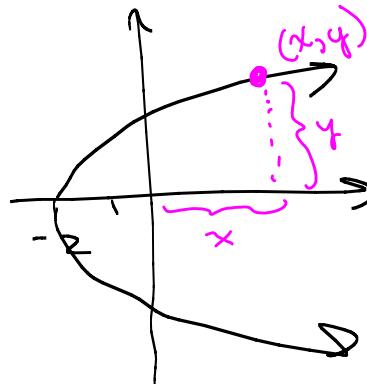
$$y^2 = 4(x+2)$$

Looks like  $y^2 = x$

$$\therefore \frac{1}{4}y^2 = x$$

$$\text{then } \frac{1}{4}y^2 = x+2$$

$$\text{Vertex: } (-2, 0)$$



Know:  $\frac{dy}{dt} = +3 \frac{\text{units}}{\text{sec}}$

Want:  $\frac{dx}{dt}$

When:  $x = 7$   
 $y = 6$

\* Need a relationship (equation) between  $x$  and  $y$ .

$$y^2 = 4x + 8$$

$$\frac{d}{dt}(y^2) = \frac{d}{dt}(4x+8)$$

$$2y \frac{dy}{dt} = 4 \frac{dx}{dt} + 0$$

$$\frac{dx}{dt} = \frac{2y \frac{dy}{dt}}{4} = \frac{y}{2} \cdot \frac{dy}{dt}$$

$x = 7, y = 6, \frac{dy}{dt} = 3 \frac{\text{units}}{\text{sec}}$

$$\frac{dx}{dt} = \frac{6}{2} \cdot 3 \frac{\text{units}}{\text{sec}}$$

$$= 9 \frac{\text{units}}{\text{sec}}$$

Implicit Diff:

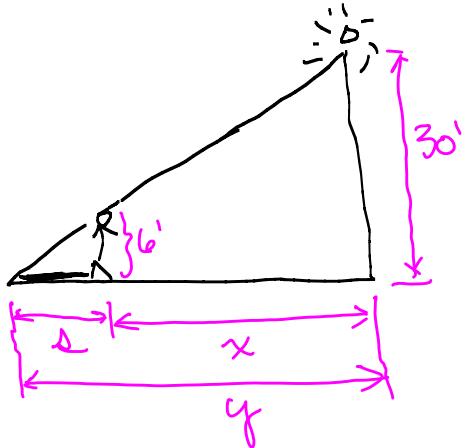
$$\frac{d}{dt}$$

$$(y^2) = \frac{d}{dt}(4x+8)$$

$$2y \frac{dy}{dt} = 4 \frac{dx}{dt} + 0$$

$$\frac{dx}{dt} = \frac{2y \frac{dy}{dt}}{4} = \frac{y}{2} \cdot \frac{dy}{dt}$$

**Example 5:** A 6-foot tall man walks away from a 30-foot tall lamppost at a speed of 400 feet per minute. When he is 50 feet away from the lamppost, at what rate is his shadow lengthening? How fast is the tip of his shadow moving?



Know:  $\frac{ds}{dt} = +400 \text{ ft/min}$

Want:  $\frac{dy}{dt}$ ,  $\frac{dy}{dt}$

When:  $s = 50 \text{ ft}$

Need relationships between  $x$  and  $s$ ,  
also between  $x$  and  $y$

Similar triangles:

$$\frac{6'}{s} = \frac{30'}{y}$$

$$6'y = 30's$$

$$y = \frac{30s}{6} = 5s$$

Note that:  $s + x = y$

$$y = 5s \Rightarrow s + x = 5s$$

$$x = 4s$$

Implicit Diff:

$$\frac{dx}{dt} = 4 \frac{ds}{dt}$$

$$\frac{400 \text{ ft}}{\text{min}} = 4 \frac{ds}{dt}$$

$$\frac{ds}{dt} = 100 \text{ ft/min}$$

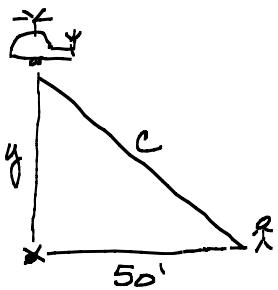
rate shadow is lengthening

$$s + x = y$$

$$\frac{ds}{dt} + \frac{dx}{dt} = \frac{dy}{dt}$$

**Example 6:** At a distance of 50 ft from the pad, a man observes a helicopter taking off from a heliport.

The helicopter is rising vertically at a speed of 44 ft/second. How fast is the distance between the helicopter and the man changing when it is at an altitude of 120 ft?



Know:  $\frac{dy}{dt} = +44 \text{ ft/sec}$

Want:  $\frac{dc}{dt}$

When:  $y = 120 \text{ ft}$

$$\frac{100 \text{ ft}}{\text{min}} + \frac{400 \text{ ft}}{\text{min}} = \frac{dy}{dt}$$

$$\frac{dy}{dt} = 500 \text{ ft/min}$$

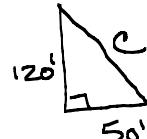
Rate tip of shadow is moving

Need to find  $c$  when  $y = 120$ .

$$\frac{dc}{dt} = \frac{y}{c} \cdot \frac{dy}{dt}$$

$$= \frac{120}{130} \cdot 44 \text{ ft/sec}$$

$$= \frac{528}{13} \approx 40.615 \text{ ft/sec}$$



$$(120^2 + 50^2) = c^2$$

$$14400 + 2500 = c^2$$

$$16900 \text{ ft}^2 = c^2$$

$$c = 130$$

$$\begin{aligned} y^2 + (50)^2 &= c^2 \\ y^2 + 2500 \text{ ft}^2 &= c^2 \\ \frac{d}{dt}(y^2 + 2500 \text{ ft}^2) &= \frac{d}{dt}(c^2) \\ 2y \frac{dy}{dt} + 0 &= 2c \frac{dc}{dt} \\ 2y \frac{dy}{dt} &= 2c \frac{dc}{dt} \end{aligned}$$

$$\frac{dc}{dt} = \frac{y}{c} \frac{dy}{dt}$$

$$\frac{dc}{dt} = \frac{y}{c} \frac{dy}{dt}$$

$$\underline{\text{Ex 2 cont'd:}} \quad V = \frac{25\pi}{432} \omega^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{25\pi}{432} \omega^3\right)$$

$$\frac{dV}{dt} = \frac{25\pi}{432} \cdot 3\omega^2 \frac{d\omega}{dt}$$

$$= \frac{75\pi}{432} \omega^2 \frac{d\omega}{dt}$$

$$\omega = 8 \text{ ft}, \frac{dV}{dt} = -\frac{10 \text{ ft}^3}{\text{min}} \Rightarrow -\frac{10 \text{ ft}^3}{\text{min}} = \frac{75\pi}{432} (8 \text{ ft})^2 \frac{d\omega}{dt}$$

$$\frac{-10 \text{ ft}^3}{\text{min}} = \frac{4800\pi \text{ ft}^2}{432} \cdot \frac{d\omega}{dt}$$

$$\frac{-10 \text{ ft}^3}{\text{min}} \cdot \frac{432}{4800\pi \text{ ft}^2} = \frac{d\omega}{dt}$$

$$\frac{d\omega}{dt} = -\frac{4320}{4800\pi} \text{ ft/min}$$

$$\approx -0.286 \text{ ft/min}$$

$$\underline{\text{Ex 3 cont'd:}} \quad \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

$$x=9, y=\sqrt{544}, \quad \left. \begin{array}{l} \frac{dx}{dt} = 2 \text{ ft/s} \\ \frac{dy}{dt} = -\frac{9}{\sqrt{544}} \cdot 2 \text{ ft/s} \end{array} \right\}$$

$$= -\frac{18}{\sqrt{544}} \text{ ft/s}$$

$$\approx -0.7117 \text{ ft/s}$$



$$\begin{aligned} a^2 + g^2 &= 2s^2 \\ a^2 &= 544 \end{aligned}$$

$$a = \sqrt{544}$$

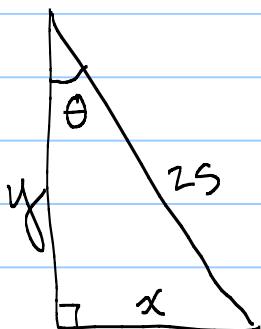


$$\begin{aligned} x &= 24 \\ a^2 + 24^2 &= 2s^2 \\ a &= 7 \text{ ft} \end{aligned}$$

$$\begin{aligned} x &= 24 \text{ ft}, y = 7 \text{ ft}, \\ \frac{dx}{dt} &= 2 \text{ ft/s} \end{aligned} \Rightarrow \frac{dy}{dt} = -\frac{24}{7} \cdot 2 \text{ ft/s}$$

$$\approx -6.857 \text{ ft/s}$$

Ex 3: Part c: Find the rate of change in the angle between the ladder and the wall when  $x = 9\text{ ft}$ . (Similar to 2.6 #21)



Know:  $\frac{dx}{dt} = +2\text{ ft/s}$

Want:  $\frac{d\theta}{dt}$

When:  $x = 9'$

Need a relationship (equation) between  $\theta$  and  $x$ :

$$\sin \theta = \frac{x}{25}$$

$$\sin \theta = \frac{1}{25} x$$

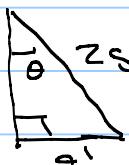
Implicit Diff:  $\frac{d}{dt}(\sin \theta) = \frac{d}{dt}\left(\frac{1}{25}x\right)$

$$(\cos \theta) \frac{d\theta}{dt} = \frac{1}{25} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{25} \cdot \frac{1}{\cos \theta} \frac{dx}{dt}$$

Note:

Find  $\theta$  when  $x = 9\text{ ft}$ :



$$\sin \theta = \frac{9}{25}$$

$$\theta = \sin^{-1}\left(\frac{9}{25}\right)$$

(for an acute angle)

From triangle Pythagoras,

$$\cos \theta = \frac{\sqrt{544}}{25}$$

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{1}{25} \cdot \frac{1}{\sqrt{544}/25} \frac{dx}{dt} \\ &= \frac{1}{\sqrt{544}} \cdot \frac{dx}{dt} \end{aligned}$$

$$= \frac{1}{\sqrt{544}} \cdot 2\text{ ft/s}$$

$$= \frac{2}{\sqrt{544}} / \text{sec}$$

inverse sin function gives back angles in the interval  $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Note:  
radian measure  $\theta = \frac{\text{arc length}}{\text{radius}}$  (unitless)

$$\approx 0.0857 / \text{sec}$$

# Homework Questions

2.5 #11  $\sin x + 2 \cos 2y = 1$ . Find  $\frac{dy}{dx}$

$$\frac{d}{dx} (\sin x + 2 \cos 2y) = \frac{d}{dx} (1)$$

$$\cos x - [2 \sin(2y)] \frac{d}{dx} (2y) = 0$$

$$\cos x - [2 \sin(2y)] (2 \frac{dy}{dx}) = 0$$

$$\cos x - 4 \frac{dy}{dx} \sin(2y) = 0$$

$$\cos x = 4 \frac{dy}{dx} \sin(2y)$$

$$\frac{\cos x}{4 \sin(2y)} = \frac{dy}{dx}$$

2.5 #25  $(x+y)^3 = x^3 + y^3$ . Find  $\frac{dy}{dx}$  at  $(-1, 1)$

$$\frac{d}{dx} (x+y)^3 = \frac{d}{dx} (x^3 + y^3)$$

$$3(x+y)^2 \frac{d}{dx} (x+y) = 3x^2 + 3y^2 \frac{dy}{dx}$$

$$3(x+y)^2 \left(1 + \frac{dy}{dx}\right) = 3x^2 + 3y^2 \frac{dy}{dx}$$

$$3(x+y)^2 + 3(x+y)^2 \frac{dy}{dx} = 3x^2 + 3y^2 \frac{dy}{dx}$$

Divide by 3:  $(x+y)^2 + (x+y)^2 \frac{dy}{dx} = x^2 + y^2 \frac{dy}{dx}$

$$(x+y)^2 \frac{dy}{dx} - y^2 \frac{dy}{dx} = x^2 - (x+y)^2$$

$$\frac{dy}{dx} ((x+y)^2 - y^2) = x^2 - (x+y)^2$$

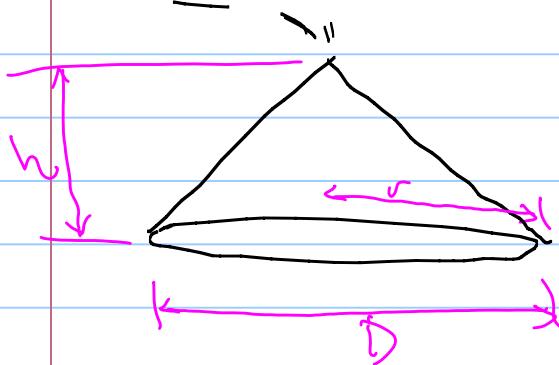
$$\frac{dy}{dx} = \frac{x^2 - (x+y)^2}{(x+y)^2 - y^2}$$

then put in  
 $x = -1$ ,  
 $y = 1$

2.6 # 17

Sand falling onto conical pile at  $10 \text{ ft}^3/\text{min}$ .  
Diameter of cone approx. 3 times its altitude.

At what rate is height changing when zile is 15' high?



Know:  $\frac{dx}{dt} = +10 \text{ ft/min}$

Want:  $\frac{dh}{dt}$

When:  $n = 15'$

$$\text{Note: } D = 3h$$

$$2r = 3h$$

$$r = \frac{3h}{2}$$

Need relationship between  $V$  and  $h$ :

$$V = \frac{1}{3} \pi r^2 h$$

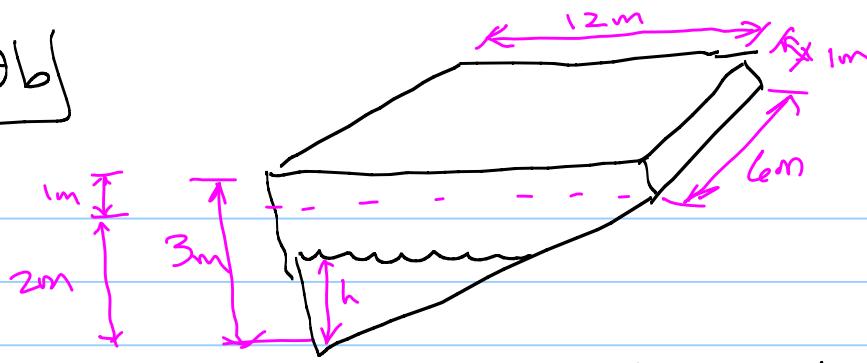
$$r = \frac{3h}{2} \Rightarrow V = \frac{1}{3}\pi \left(\frac{3h}{2}\right)^2 h$$

$$V = \frac{1}{3}\pi r^2 h$$

$$J = \frac{3\pi}{4} h^3$$

Then differentiate w/ respect to t

2.6 #19b



Water pumped in at  $\frac{1}{4} \text{ m}^3/\text{min}$ . 1 m of water at deep end.

- (a) what % of pool is filled?
- (b) at what rate is water level rising?

Know:  $\frac{dV}{dt} = \frac{1}{4} \text{ m}^3/\text{min}$

Want:  $\frac{dh}{dt}$

when:  $h = 1 \text{ m}$

Need Relationship between V and h:

(Area of triangular cross-section)(6m) = V of water



$$\frac{h}{x} = \frac{2}{12} \quad \text{or} \quad \frac{h}{2} = \frac{x}{12}$$

$$\text{Area}_{\Delta} = \frac{1}{2} (\text{base})(\text{ht})$$

$$= \frac{1}{2} (x)(h)$$

$$= \frac{1}{2} (6h)(h)$$

$$\frac{12h}{2} = x$$

$$6h = x$$

Area of water part of  $\Delta$  cross section

$$V = 3h^2 (6 \text{ m})$$