

3.1: Extrema on an Interval

(Extrema  
= Extreme Values)

Absolute maximum and minimum:

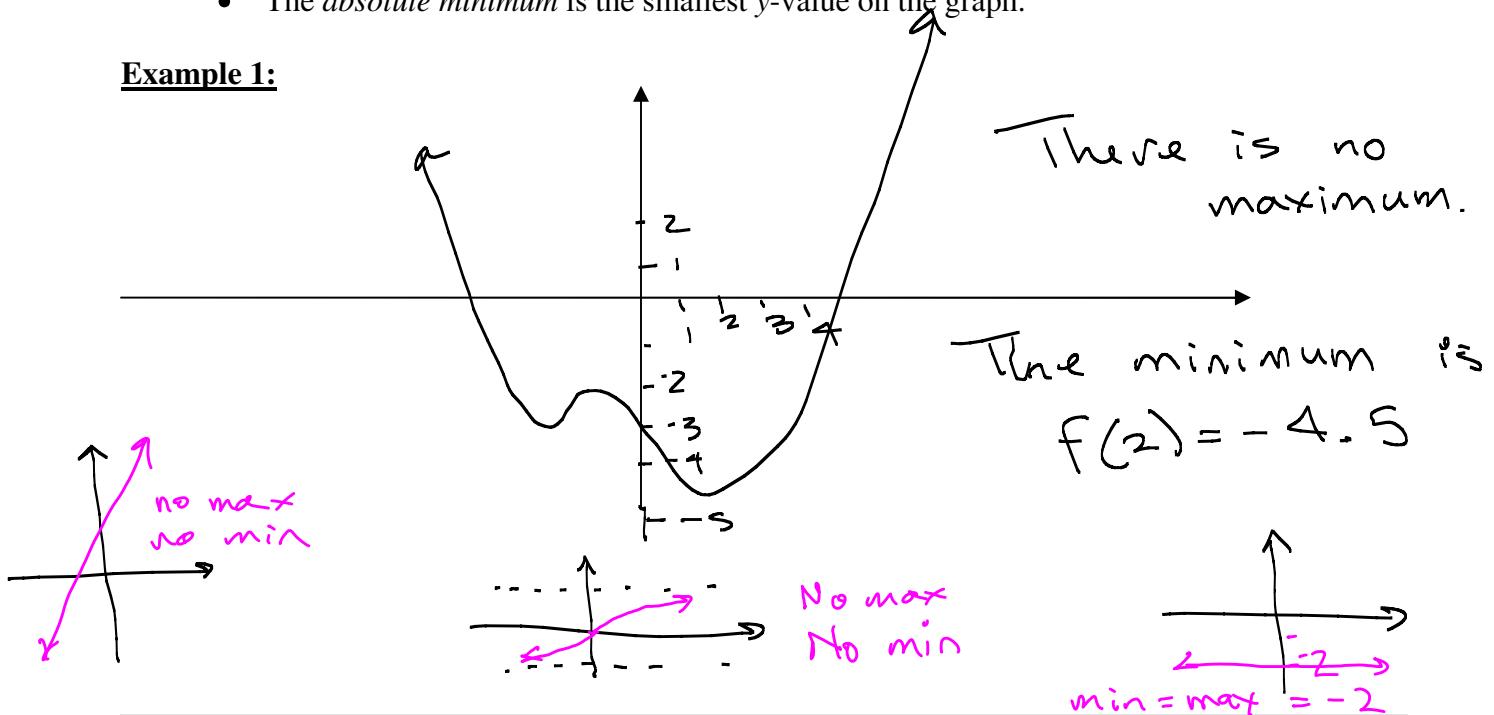
If  $f(x) \leq f(c)$  for every  $x$  in the domain of  $f$ , then  $f(c)$  is the *maximum*, or *absolute maximum*, of  $f$ .

If  $f(x) \geq f(c)$  for every  $x$  in the domain of  $f$ , then  $f(c)$  is the *minimum*, or *absolute minimum* of  $f$ .

The maximum and minimum values of a function are called the *extreme values* of the function.

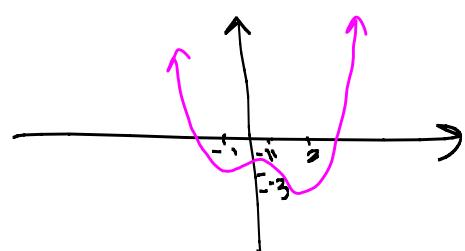
In other words,

- The *absolute maximum* is the largest  $y$ -value on the graph.
- The *absolute minimum* is the smallest  $y$ -value on the graph.

Example 1:

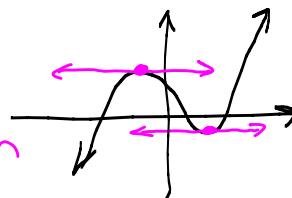
Relative (Local) Maxima and Minima:

- A function  $f$  has a *relative maximum*, or *local maximum*, at  $x=c$  if there is an interval  $(a,b)$  around  $c$  such that  $f(x) \leq f(c)$  for every  $x$  in  $(a,b)$ . (These are the “hilltops”).
- A function  $f$  has a *relative minimum*, or *local minimum*, at  $x=c$  if there is an interval  $(a,b)$  around  $c$  such that  $f(x) \geq f(c)$  for every  $x$  in  $(a,b)$ . (These are the “bottoms of valleys”).



Relative max:  $f(0.5) = -1$   
Relative minima:  $f(-1) \approx -2$   
 $f(1.5) \approx -3$

horizontal tangent lies at relative max and rel. min



No absolute max  
No absolute min

3.1.2

Notice: If the function is differentiable (smooth), then the tangent line at a local minimum or maximum is horizontal.

(relative)

Fermat's Theorem: If  $f$  has a local maximum or minimum at  $c$ , and if  $f'(c)$  exists, then  $f'(c) = 0$ .

This means that if  $f$  is differentiable at  $c$  and has a relative extreme at  $c$ , then the tangent line to  $f$  at  $c$  must be horizontal.

However, we must be careful. The fact that  $f'(c) = 0$  (tangent line is horizontal) does not guarantee that there is a relative minimum or maximum at  $c$ .

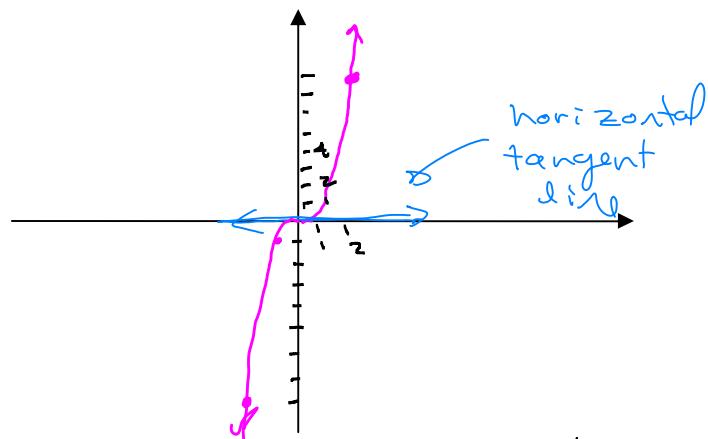
Example 2:  $f(x) = x^3$

$$f'(x) = 3x^2$$

Where is the derivative 0?  
at  $x=0$ .

$$f'(0) = 3(0)^2 = 0$$

$$\begin{array}{|c|c|} \hline x & x^3 \\ \hline -1 & -1 \\ 0 & 0 \\ 1 & 1 \\ \hline \end{array}$$



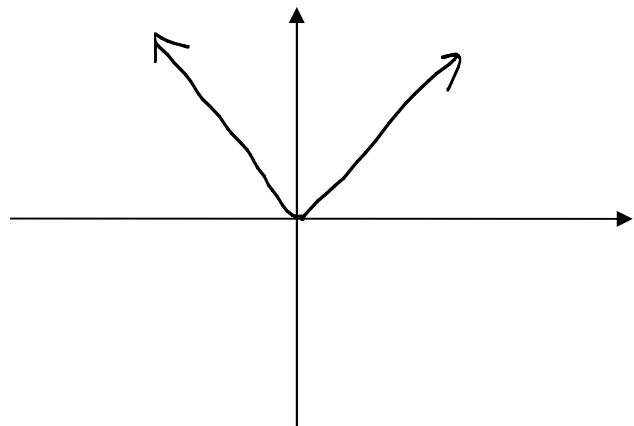
No absolute or relative max or min.

Example 3: There can be a local maximum or minimum at  $c$  even if  $f'(c)$  does not exist.

$$f(x) = |x|$$

Relative minimum:  $f(0) = 0$

$f$  is not differentiable at 0.



Ex:  $f(x) = \tan x$

$$f'(x) = \sec^2 x$$

$$f'(0) = (\sec(0))^2$$

$$= \left(\frac{1}{\cos 0}\right)^2$$

$$= (1)^2$$

$$= 1 = \text{slope}$$

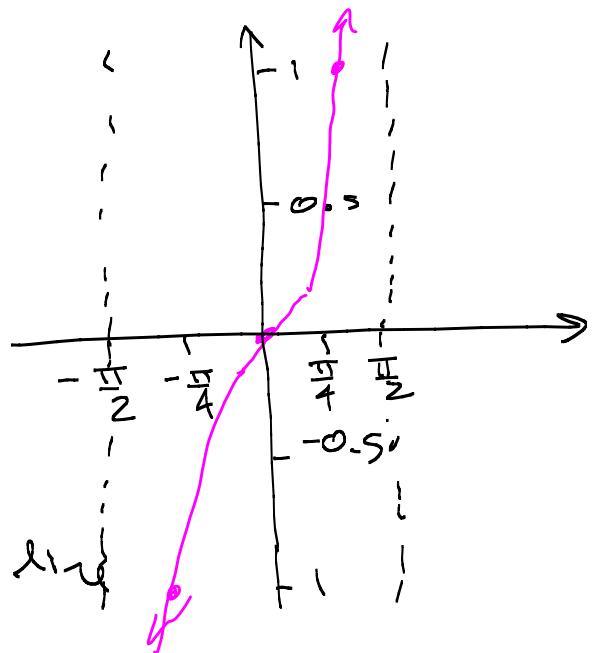
of tangent line

Note:  $f'(\frac{\pi}{4}) = (\sec \frac{\pi}{4})^2$

$$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \quad = \left(\frac{1}{\cos \frac{\pi}{4}}\right)^2$$

$$\sec \frac{\pi}{4} = \sqrt{2} \quad = (\sqrt{2})^2 \\ = 2$$

$$f(\frac{\pi}{4}) = \tan \frac{\pi}{4} = 1$$



**Critical numbers:**

**Critical Number:** A *critical number* of a function  $f$  is a number  $c$  in the domain of  $f$  such that either  $f'(c) = 0$  or  $f'(c)$  does not exist.

**Theorem:** If  $f$  has a local maximum or minimum at  $c$ , then  $c$  is a critical number of  $f$ .

**Note:** The converse of this theorem is not true. It is possible for  $f$  to have a critical number at  $c$ , but not to have a local maximum or minimum at  $c$ .

**Example 4:** Find the critical numbers of  $f(x) = x^3 + \frac{17}{2}x^2 - 6x + 4$ .

Find derivative:  $f'(x) = 3x^2 + \frac{17}{2}(2x) - 6$   
 $= 3x^2 + 17x - 6$

Set  $f'(x) = 0$ :  $0 = 3x^2 + 17x - 6$   
 $0 = (3x - 1)(x + 6)$

$$\begin{aligned} 3x - 1 &= 0 & x + 6 &= 0 \\ 3x &= 1 & x &= -6 \\ x &= \frac{1}{3} \end{aligned}$$

Critical Numbers:  $\frac{1}{3}, -6$

**Example 5:** Find the critical numbers of  $f(x) = x^{2/3}$

$$f(x) = x^{2/3}$$

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

Critical number: 0

Note:  
 $f(x) = x^{2/3} = \sqrt[3]{x^2}$   
Domain:  $(-\infty, \infty)$

The numerator is never 0.  
The denominator is 0 when  $x=0$ .  
So  $f'(0)$  is not defined (does not exist)

**Example 6:** Find the critical numbers of  $f(x) = \frac{x^2}{x-3}$

Domain of  $f$ :  
 $x \neq 3$   
 $(-\infty, 3) \cup (3, \infty)$

$$\begin{aligned} f'(x) &= \frac{(x-3)\frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(x-3)}{(x-3)^2} \\ &= \frac{(x-3)(2x) - x^2(1)}{(x-3)^2} = \frac{2x^2 - 6x - x^2}{(x-3)^2} \\ &= \frac{x^2 - 6x}{(x-3)^2} = \frac{x(x-6)}{(x-3)^2} \end{aligned}$$

Set  $f'(x) = 0$ :  $\frac{x(x-6)}{(x-3)^2} = 0$ . Numerator is 0 for  $x=0, x=6$   
Denominator is 0 for  $x=3$

So  $f'(0)=0, f'(6)=0, f'(3)$  doesn't exist. But 3 is not in the

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domain of  $f$ , so it can't be a critical number.

Critical numbers: 0, 6

3.1.4

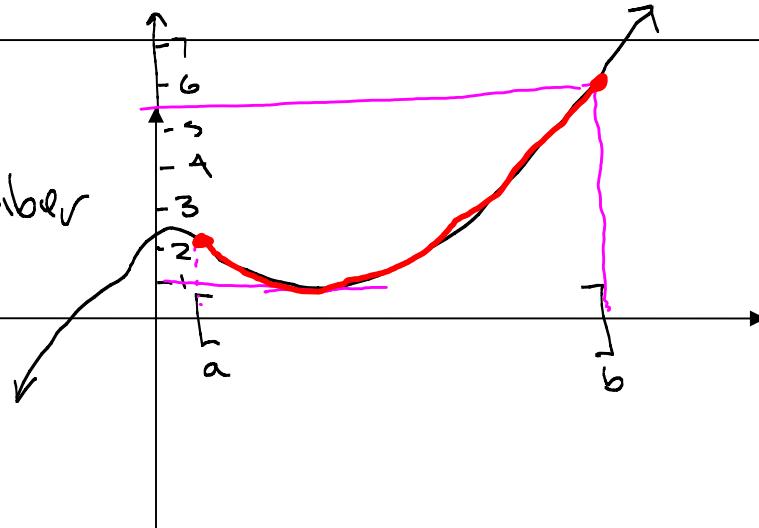
### Absolute extrema on a closed interval:

Extreme Value Theorem: If  $f$  is continuous on a closed interval  $[a, b]$ , then  $f$  has both an absolute maximum and an absolute minimum on  $[a, b]$ .

Note: The absolute maximum and the absolute minimum must occur at either a critical value in  $(a, b)$  or at an endpoint (at  $a$  or  $b$ ).

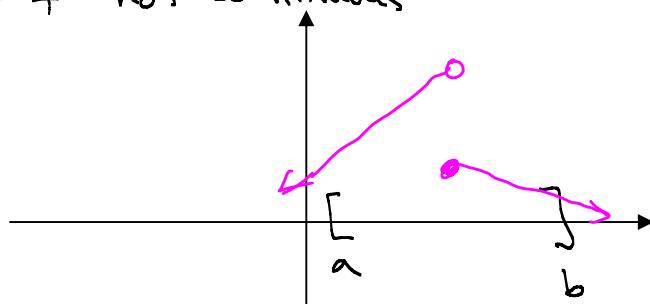
#### Example 7:

Absolute min  $\approx 1$   
occurs at a critical number  
Absolute max:  
 $\approx 5.5 - 6$   
(occurs at an endpoint)

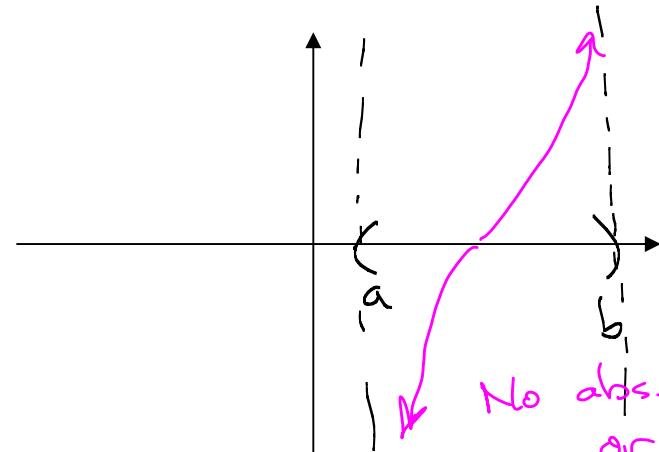


Example 8: If either hypothesis of the extreme value theorem (continuity or closed interval) is violated, the existence of an absolute maximum or minimum is not guaranteed.

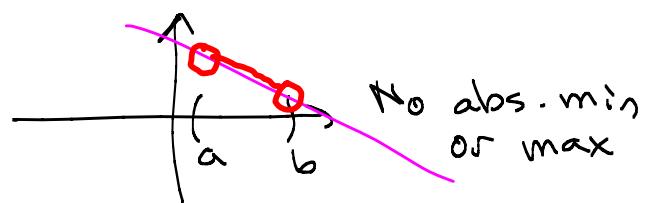
IF  $f$  not continuous



Does not have an absolute max.



No abs. min or max



No abs. min or max

Step 0: Check that  $f$  is continuous.

Process for Finding the Absolute Extrema of a Continuous Function on a Closed Interval:

1. Find the critical values in  $(a, b)$ .
2. Compute the value of  $f$  at each critical value in  $(a, b)$  and also compute  $f(a)$  and  $f(b)$ .
3. The absolute maximum is the largest of these  $y$ -values and the absolute minimum is the smallest of these  $y$ -values.

Example 9: Find the absolute extrema for  $f(x) = x^2 + 2$  on the interval  $[-2, 3]$ .

Step 0: Is  $f$  continuous? Yes.

Step 1: Find critical numbers:  $f'(x) = 2x = 0$

$$x = 0$$

Critical number: 0

Step 2: Find  $y$ -values at critical #s and endpoints.

$$f(0) = 0^2 + 2 = 2$$

$$f(-2) = (-2)^2 + 2 = 4 + 2 = 6$$

$$f(3) = (3)^2 + 2 = 9 + 2 = 11$$

$$\text{Absolute max: } f(3) = 11$$

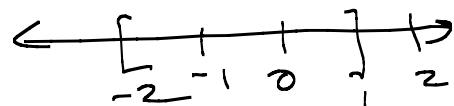
$$\text{Absolute min: } f(0) = 2$$

Example 10: Find the extreme values of  $g(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 - 2x^2 + 3$  on the interval  $[-2, 1]$ .

$$\begin{aligned} g'(x) &= \frac{1}{2}(4x^3) - \frac{2}{3}(3x^2) - 4x \\ &= 2x^3 - 2x^2 - 4x \\ &= 2x(x^2 - x - 2) \\ &= 2x(x-2)(x+1) \end{aligned}$$

$$g'(x) = 0 \therefore x = 0, 2, -1$$

2 is not in interval



$$g(0) = 3$$

$$g(-1) = \frac{1}{2}(-1)^4 - \frac{2}{3}(-1)^3 - 2(-1)^2 + 3$$

$$= \frac{1}{2} + \frac{2}{3} - 2 + 3$$

$$= \frac{3}{6} + \frac{4}{6} + \frac{6}{6} = \frac{13}{6} = 2\frac{1}{6}$$

$$g(-2) = \frac{1}{2}(-2)^4 - \frac{2}{3}(-2)^3 - 2(-2)^2 + 3$$

$$= \frac{1}{2}(16) - \frac{2}{3}(-8) - 8 + 3 \quad , \frac{8}{3}$$

$$= 8 + \frac{16}{3} - 8 + 3 = \frac{16}{3} + \frac{9}{3} = \frac{25}{3}$$

$$g(1) = \frac{1}{2}(1)^4 - \frac{2}{3}(1)^3 - 2(1)^2 + 3$$

$$= \frac{3}{6} - \frac{4}{6} - 2 + 3$$

$$= -\frac{1}{6} + 1 = \frac{5}{6}$$

smallest

$$\text{Absolute max: } g(-2) = \frac{25}{3}$$

$$\text{Absolute min: } g(1) = \frac{5}{6}$$

**Example 11:** Find the absolute extrema of  $h(x) = 6x^{\frac{2}{3}}$  on the intervals (a)  $[-8, 1]$ , (b)  $[-8, 1)$ , and (c)  $(-8, 1)$ .

$$h(x) = 6x^{\frac{2}{3}} = 6\sqrt[3]{x^2}$$

$h$  is continuous on  $(-\infty, \infty)$

Find critical #s:  $h'(x) = 6\left(\frac{2}{3}x^{-\frac{1}{3}}\right) = 4x^{-\frac{1}{3}} = \frac{4}{\sqrt[3]{x}}$   
Derivative does not exist at  $x=0$  (where denominator is 0)

Numerator is never 0.

Only critical number is 0. Note that  $0 \in [-8, 1]$

Find y-values:  $h(0) = 6\sqrt[3]{0^2} = 6 \cdot 0 = 0$   
 $h(-8) = 6\sqrt[3]{(-8)^2} = 6\sqrt[3]{64} = 6(4) = 24$   
 $h(1) = 6\sqrt[3]{1^2} = 6(1) = 6$

(a) Absolute min is  $h(0) = 0$ ; absolute max is  $h(-8) = 24$ .

**Example 12:** Find the absolute maximum and absolute minimum of  $f(x) = \sin 2x - x$  on the interval  $[0, \pi]$ .

$$\begin{aligned} f'(x) &= (\cos 2x)(2) - 1 \\ &= 2\cos 2x - 1 \end{aligned}$$

$$\text{Set } f'(x) = 0: 0 = 2\cos 2x - 1$$

$$1 = 2\cos 2x$$

$$\frac{1}{2} = \cos 2x$$

$$\cos 2x = \frac{1}{2}$$

(It might help to let  $\theta = 2x$   
then  $\cos \theta = \frac{1}{2}$ )

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3} + 2k\pi$$

$$\text{On } [0, 2\pi], 2x = \frac{\pi}{3}, \frac{5\pi}{3} + 2k\pi$$

$$\text{Divide by 2: } x = \frac{\pi}{6} + \frac{2k\pi}{2}, \frac{5\pi}{6} + \frac{2k\pi}{2}$$

$$x = \frac{\pi}{6} + k\pi, \frac{5\pi}{6} + k\pi$$

Only critical numbers in interval are  $\frac{\pi}{6}, \frac{5\pi}{6}$ .

$$f(0) = \sin(2(0)) - 0 = 0$$

$$f(\pi) = \sin(2\pi) - \pi = 0 - \pi = -\pi \approx -3.14$$

$$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \approx 0.343$$

$$\begin{aligned} f\left(\frac{5\pi}{6}\right) &= \sin\left(2 \cdot \frac{5\pi}{6}\right) - \frac{5\pi}{6} = \sin\left(\frac{5\pi}{3}\right) - \frac{5\pi}{6} \\ &= -\frac{\sqrt{3}}{2} - \frac{5\pi}{6} \approx -3.48 \end{aligned}$$

$$\text{Absolute max: } f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

$$\text{Absolute min: } f\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} - \frac{5\pi}{6}$$

Ex 11 cont'd.

⑥  $[-8, 1)$

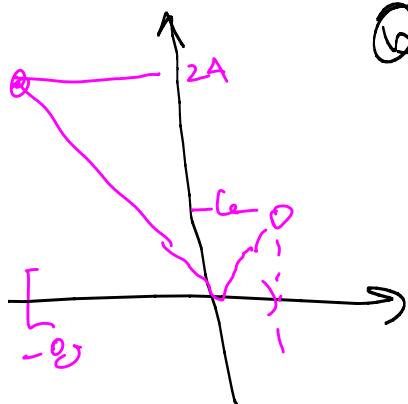
Extreme value Theorem does not apply.

$$h(x) = 6\sqrt[3]{x^2}$$

$$h'(x) = \frac{4}{3\sqrt{x}}$$

for  $x > 0$ ,  $h'(x) > 0$ . So tangent lines have positive slope.

For  $x < 0$ ,  $h'(x) < 0$ . So tangent lines have negative slopes.



⑥ On  $[-8, 1)$

Absolute min:  $h(0) = 0$

Absolute max:  $h(-8) = 24$

⑦ on  $(-8, 1)$

Absolute min:  $h(0) = 0$

No absolute max.