

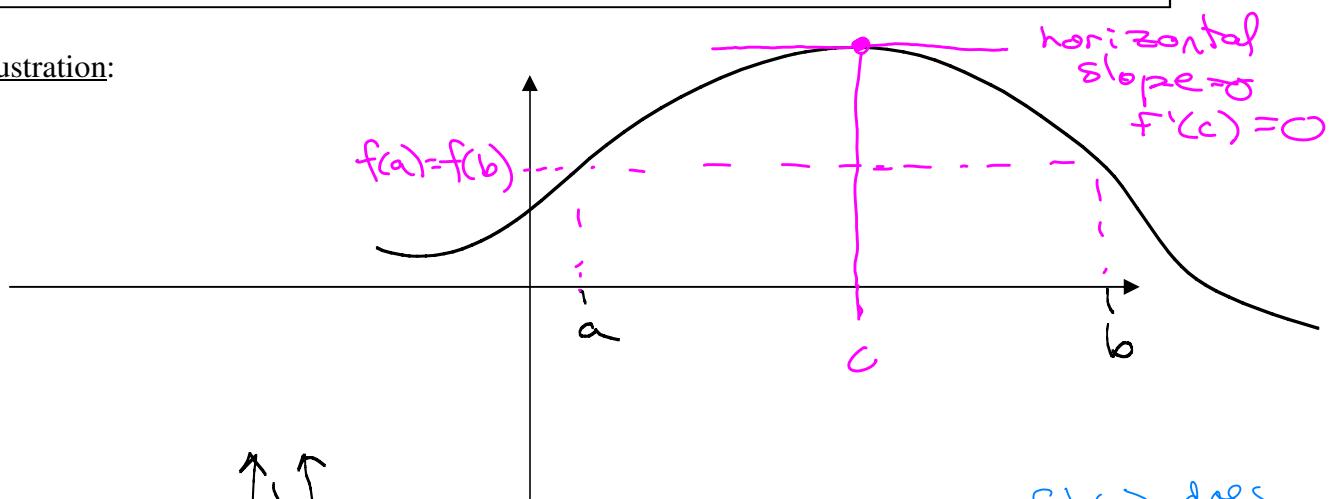
### 3.2: Rolle's Theorem and the Mean Value Theorem

#### Rolle's Theorem:

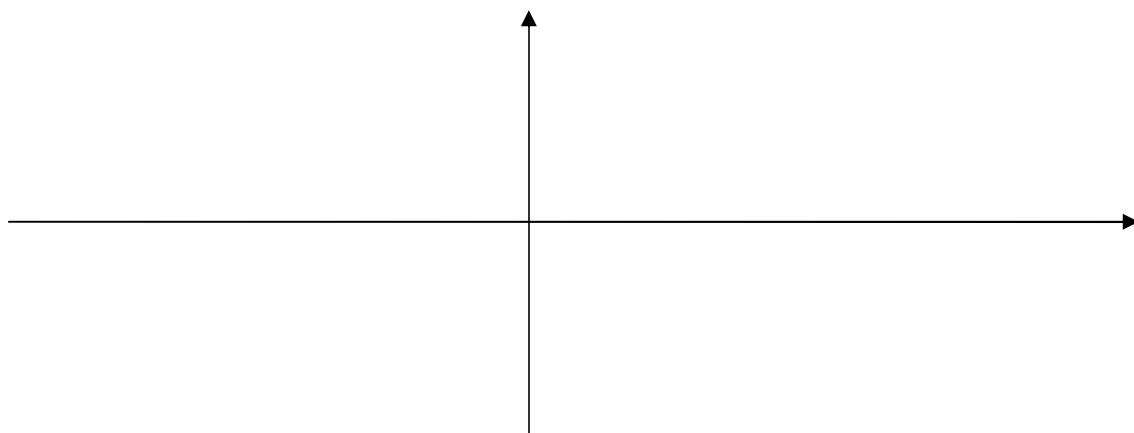
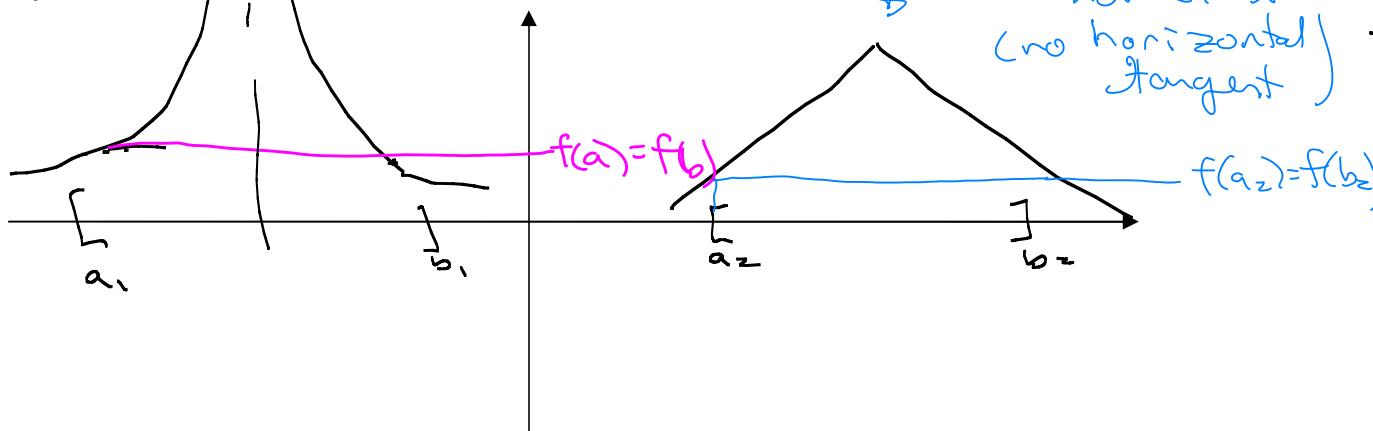
Let  $f$  be a function that is continuous on the closed interval  $[a,b]$  and differentiable on the open interval  $(a,b)$ .

If  $f(a) = f(b)$ , then there is a number  $c$  in  $(a,b)$  such that  $f'(c) = 0$ .

#### Illustration:



What if  $f$  is not continuous? Not differentiable?



**Example 1:** Show that the function  $f(x) = x^2 - 4x - 5$  satisfies the hypotheses of Rolle's Theorem on the interval  $[-1, 5]$ . Find all numbers  $c$  in  $[-1, 5]$  that satisfy the conclusion of Rolle's Theorem.

Hypotheses of Rolle's:  $f$  is continuous on  $[-1, 5]$ ? Yes.  
 $f$  is differentiable on  $(-1, 5)$ ? Yes.  
 (Polynomials are continuous and differentiable on  $(-\infty, \infty)$ )

Show  $f(a) = f(b)$ :

Show  $f(-1) = f(5)$ :

$$\begin{aligned} f(-1) &= (-1)^2 - 4(-1) - 5 = 1 + 4 - 5 = 0 \\ f(5) &= 5^2 - 4(5) - 5 = 25 - 20 - 5 = 0 \end{aligned}$$

$$f'(x) = 2x - 4 \quad \rightarrow 2x - 4 = 0 \quad \text{so } c = 2 \text{ satisfies} \\ \text{Set } f'(x) = 0: \quad \begin{array}{l} 2x = 4 \\ x = 2 \end{array} \quad \text{the conclusion.}$$

**Example 2:** Show that the function  $g(x) = -2x^4 + 16x^2$  satisfies the hypotheses of Rolle's Theorem on the interval  $[-3, 3]$ . Find all numbers  $c$  that satisfy the conclusion of Rolle's Theorem.

Continuous, differentiable on  $(-\infty, \infty)$

$g(-3) = g(3)$  because powers are even.

$$g(x) = -2x^4 + 16x^2$$

$$g'(x) = -8x^3 + 32x$$

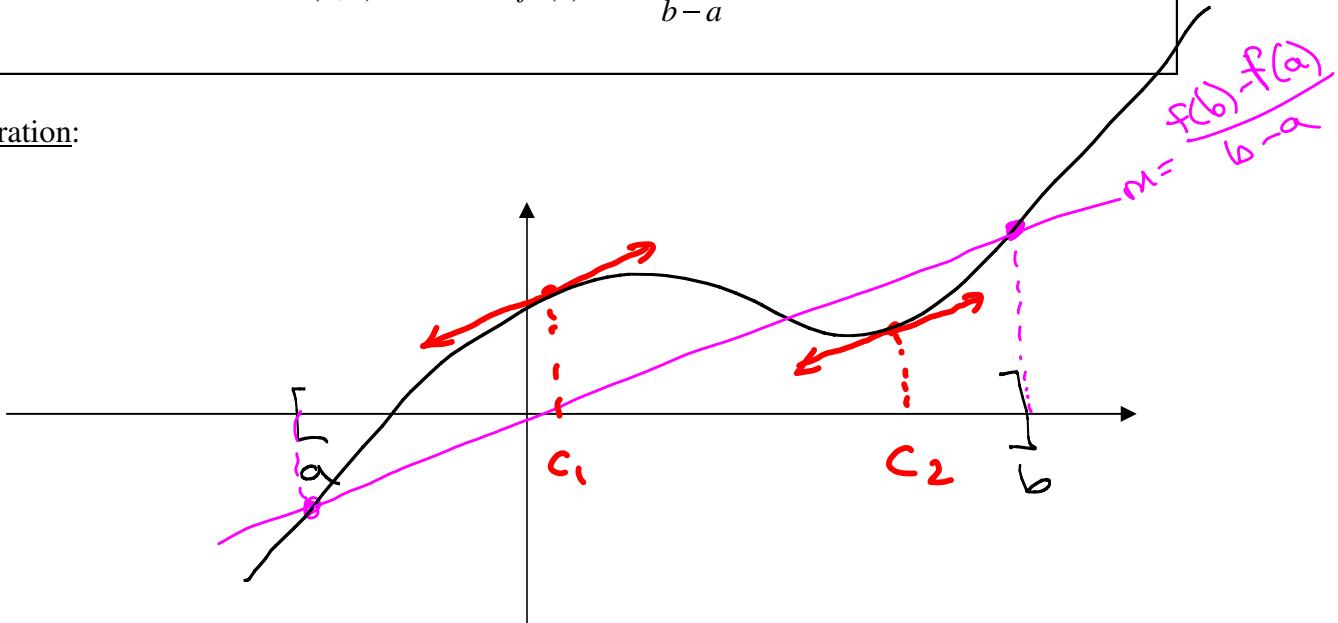
$$\begin{aligned} \text{Set } g'(x) = 0: \quad 0 &= -8x^3 + 32x \\ 0 &= -8x(x^2 - 4) \\ 0 &= -8x(x+2)(x-2) \\ x &= 0, -2, 2 \end{aligned}$$

0, -2, 2 are  
the  $c$ -values

Mean Value Theorem:

Let  $f$  be a function that is continuous on the closed interval  $[a,b]$  and differentiable on the open interval  $(a,b)$ .

Then there is a number  $c$  in  $(a,b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$ .

Illustration:

A few consequences of the Mean Value Theorem:

- 1) The Mean Value Theorem guarantees the existence of a tangent line parallel to the secant line that contains the endpoints  $(a, f(a))$  and  $(b, f(b))$ .
- 2) In terms of rates of change, the Mean Value Theorem guarantees that there is some point at which the instantaneous rate of change is equal to the average rate of change over  $[a,b]$ .
- 3) If the derivative of a function is 0 for every number in an interval, then the function is constant on that interval.
- 4) If two functions have the same derivative on an interval, they differ by a constant on that interval.

**Example 3:** Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers  $c$  that satisfy the conclusion of the Mean Value Theorem.

$$f(x) = \sqrt{x} - 2x \text{ on the interval } [0, 4]$$

$$f(x) = x^{\frac{1}{2}} - 2x$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 2$$

$$= \frac{1}{2\sqrt{x}} - 2$$

$$\text{Domain: } x \geq 0$$

$$[0, \infty)$$

continuous on  $[0, \infty)$

so continuous on  $[0, 4]$

$f'(x)$  undefined at 0.

But differentiable on  $(0, \infty)$  and thus  $(0, 4)$ .

Find slope of secant line:

cont'd at end of notes.

**Example 4:** As you drive by a Houston police car, your speed is clocked at 50 miles per hour. Five minutes and five miles further, you pass by another HPD officer, who clocks you at 55 miles per hour. If the speed limit on the entire stretch of highway is 55 mph, would the second officer be justified in writing you a speeding ticket?

5 miles in 5 minutes  $\Rightarrow$  1 mile/minute

= 60 miles/hour

= average velocity for that  
5-minute interval

From MVT, (Mean Value Thm), my instantaneous velocity had to be 60 miles/hr somewhere in that 5-minute interval.

So yes, the ticket is justified.

# Homework Questions

3.1 # 40  $f(x) = \sqrt{4-x^2}$ . Find max & min

- on (a)  $[-2, -2]$ , (b)  $[-2, 0)$ , (c)  $(-2, 2)$   
 (d)  $[1, 2)$

$$f(x) = (4-x^2)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{4-x^2}}$$

Domain of  $f(x) = \sqrt{4-x^2}$

$$4-x^2 \geq 0$$

$$(2+x)(2-x) \geq 0$$

Domain of:  $[-2, 2]$

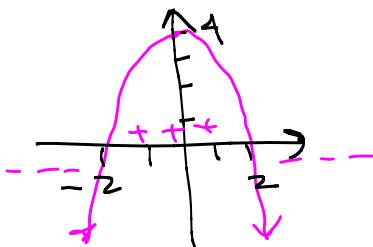
think of  
 $y = 4-x^2$   
 $y = -x^2 + 4$

Find x-intercepts: set  $y=0$ :  $0 = -x^2 + 4$

$$0 = -(x^2 - 4)$$

$$0 = -(x+2)(x-2)$$

$$x = \pm 2$$



$$\text{so } y = -x^2 + 4 \\ = 4 - x^2$$

is  $\geq 0$  for

$$-2 \leq x \leq 2$$

$$f(x) = \sqrt{4-x^2}$$

$$f'(x) = \frac{-x}{\sqrt{4-x^2}}$$

Find critical numbers:  $f'(x) = 0$  for  $x=0$  (when numerator is 0)

Where is  $f'(x)$  undefined? at  $x=2, x=-2$  (where denominator is 0)  
 Critical #:  $0, 2, -2$

For (a) Find  $f(0), f(2), f(-2)$ .

$$f(0) = \sqrt{4-0^2} = 2$$

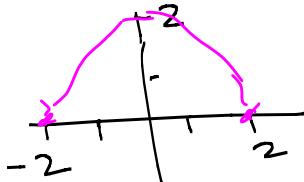
Absolute max on  $[-2, 2]$  is  $f(0)=2$ .

$$f(2) = \sqrt{4-2^2} = 0$$

Absolute min on  $[-2, 2]$  is  
 $f(2) = f(-2) = 0$ .

$$f(-2) =$$

(b)

on  $[-2, 0]$ 

No absolute max.

Absolute min:  $f(-2) = 0$ (c) On  $(-2, 2)$ Absolute max:  $f(0) = 2$   
No absolute min(d) On  $[1, 2]$ , absolute max  $\rightarrow f(1) = \sqrt{3}$   
No absolute min3.2 Example 3 cont'd:  $f(x) = \sqrt{x} - 2x$  on  $[0, 4]$ 

$$f'(x) = \frac{1}{2\sqrt{x}} - 2$$

Find slope of secant line:

$$f(0) = \sqrt{0} - 2(0) = 0$$

$$f(4) = \sqrt{4} - 2(4) = 2 - 8 = -6$$

$$\text{Slope: } m = \frac{-6 - 0}{4 - 0} = -\frac{3}{2}$$

$$\text{Set } f'(x) = -\frac{3}{2}: \quad \frac{1}{2\sqrt{x}} - 2 = -\frac{3}{2}$$

$$\text{Multiply by } 2\sqrt{x}: \quad 1 - 4\sqrt{x} = -3\sqrt{x}$$

$$1 = 4\sqrt{x} - 3\sqrt{x}$$

$$1 = \sqrt{x}$$

$$\text{Square both sides: } (1)^2 = (\sqrt{x})^2$$

$$x = 1$$

So  $c = 1$  is the required value that satisfies the MVT.

Check:

$$f'(x) = \frac{1}{2\sqrt{x}} - 2$$

$$f'(1) = \frac{1}{2\sqrt{1}} - 2$$

$$= \frac{1}{2} - 2$$

$$= \frac{1}{2} - \frac{4}{2} = -\frac{3}{2}$$

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