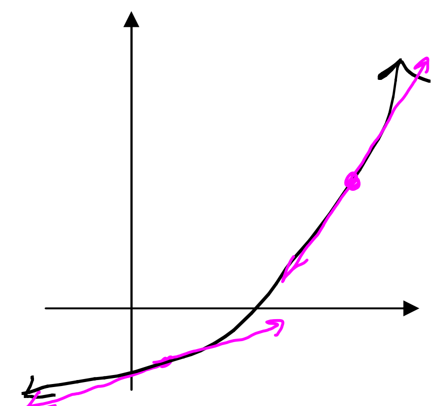
3.4: Concavity and the Second Derivative Test

Concavity:

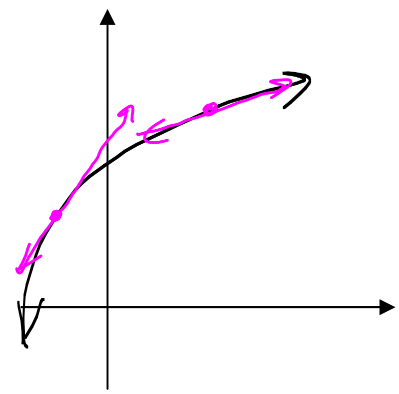
Definition:

- If the graph of f lies above all of its tangents on an interval, then it is called *concave upward* on that interval. (below its secant lines)
- If the graph of f lies below all its tangents on an interval, it is called *concave downward* on that interval. (above its secant lines)

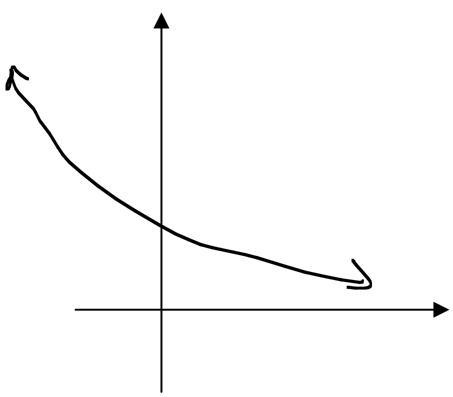
Illustration:



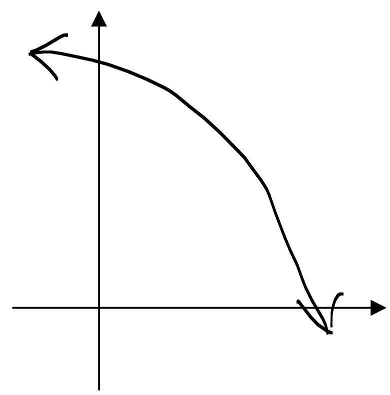
Increasing and Concave Up



Increasing and Concave Down



Decreasing and Concave Up



Decreasing and Concave Down

Notice the slopes of the tangent lines. When the curve is concave up, the slopes are increasing as you move from left to right.

When the curve is concave down, the slopes are decreasing as you move from left to right.

We find out whether f' is increasing or decreasing by looking at its derivative, which is f''.

Concavity Test:

- If $f''(x) > 0$ for all x in (a,b), then *f* is concave up on (a,b).
- If $f''(x) < 0$ for all x in (a,b), then *f* is concave down on (a,b).

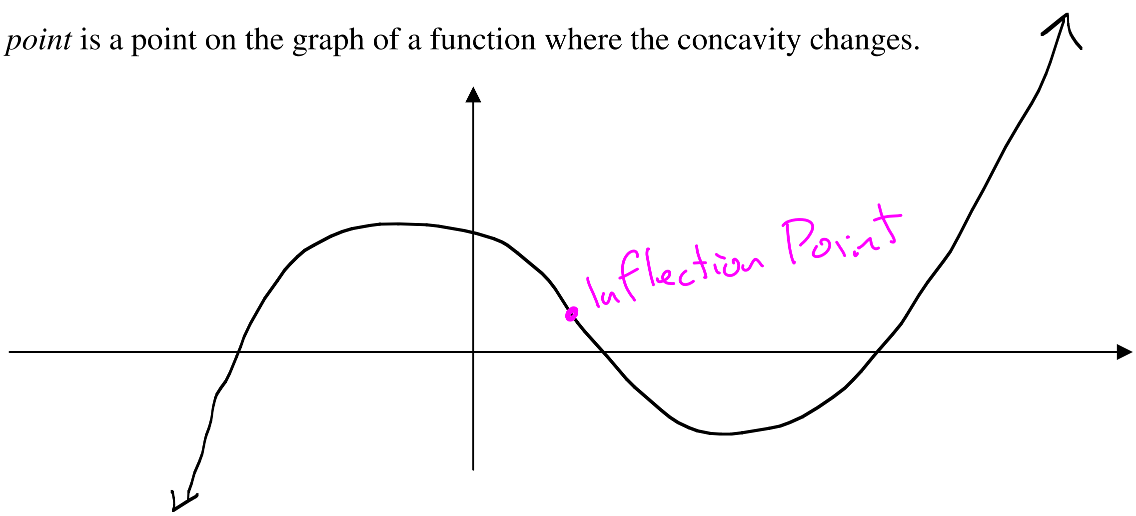
Process for Determining Intervals of Concavity:

1. Find the values of x where $f''(x) = 0$ or where $f''(x)$ is not defined. Use these values of x to divide the number line into intervals. (Numbers where f is undefined must also be on number line)
2. Choose a test number c in each interval.
   - If $f''(c) > 0$, then *f* is concave up on that interval.
   - If $f''(c) < 0$, then *f* is concave down on that interval.

Inflection points:

An *inflection point* is a point on the graph of a function where the concavity changes.

Example 1:



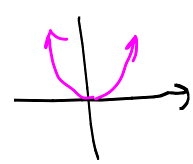
Example 2: Find the intervals on which $f(x) = x^2$ is concave up and concave down.

$f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2 > 0$ for all x

So f is concave up on $(-\infty, \infty)$



Example 3: Determine the intervals of concavity and the inflection points of $f(x) = x^3 + 6x^2 - 36x + 18$.

$f'(x) = 3x^2 + 12x - 36$

$f''(x) = 6x + 12$

$= 6(x+2)$

Where is $f''(x) = 0$? At $x = -2$

$(-\infty, -2)$: Test $x = -3$

$f''(-3) = 6(-3) + 12$

$= -18 + 12 = -6$ $(-)$

$(2, \infty)$: Test $x = 0$

$f''(0) = 6(0) + 12 = 12$ $(+)$

f'': $- - - -$ | $+ + +$ at -2

concave down | concave up

Concave down on $(-\infty, -2)$.

Concave up on $(2, \infty)$.

Inflection Point: $(-2, 106)$

Find y-value:

$f(-2) = (-2)^3 + 6(-2)^2 - 36(-2) + 18$

$= -8 + 24 + 72 + 18 = 16 + 90$

$= 106$

Example 4: Determine the intervals of concavity and the inflection points of $f(x) = x + \frac{4}{x}$.

$= x + 4x^{-1}$

$f'(x) = 1 - 4x^{-2}$

$f''(x) = 8x^{-3} = \frac{8}{x^3}$

f'': $- - -$ ¦ $+ +$ (f ND)

conc down | conc up

f'' is positive for $x > 0$

f'' is negative for $x < 0$

$$\boxed{\begin{array}{l}\text{Concave down on } (-\infty, 0).\\ \text{Concave up on } (0, \infty).\\ \text{No inflection points}\end{array}}$$

The second derivative test:

Notice: For a smooth (differentiable) function, the graph is concave upward at a relative minimum and concave downward at a relative maximum.

Therefore, at a critical number, we can look at the sign of f'' to determine whether there is a relative minimum or relative maximum at that critical number.

> The Second Derivative Test (for Local Extremes):
>
> Suppose f'' is continuous near c.
> - If $f'(x)=0$ and $f''(c)<0$, then f has a relative maximum at c.
> - If $f'(x)=0$ and $f''(c)>0$, then f has a relative minimum at c.
> - If $f'(x)=0$ and $f''(c)=0$, then the test is inconclusive. Use the 1st derivative test instead.

Example 5: Use the second derivative test to find the local extremes of $f(x)=x^3+6x^2-36x+18$.

Find critical numbers (candidates): $f'(x)=3x^2+12x-36$

Set $f'(x)=0$: $0=3(x^2+4x-12)$

$0=3(x+6)(x-2)$

Critical #s: $x=-6$, $x=2$

$f''(x)=6x+12$

Plug critical numbers into $f''(x)$:

$f''(-6)=6(-6)+12$

$=-24$ $(-)$ concave down

$\boxed{\text{Relative max at } x=-6}$

$f''(2)=6(2)+12$

$(+)$

concave up

$\boxed{\text{Relative min at } x=2}$

(should find y-values)

Example 6: Determine the local extremes of $f(x)=-2x^4+4x^3$

$f(x)=-2x^4+4x^3$

Find critical #s (candidates):

$f'(x)=-8x^3+12x^2$

$=-4x^2(2x-3)$

Critical Numbers: $0, \frac{3}{2}$

$f''(x)=-24x^2+24x$

Put critical numbers into $f''(x)$: (2nd derivative test)

$f''(0) = -24(0)^2 + 24(0) = 0$ 2nd derivative test is inconclusive. Need to use 1st derivative test

$$f''\left(\frac{3}{2}\right) = -24\left(\frac{3}{2}\right)^2 + \overset{12}{\cancel{24}}\left(\frac{3}{\underset{1}{\cancel{2}}}\right)$$

$$= -\overset{6}{\cancel{24}}\left(\frac{9}{\underset{1}{\cancel{4}}}\right) + 36$$

$$= -54 + 36$$

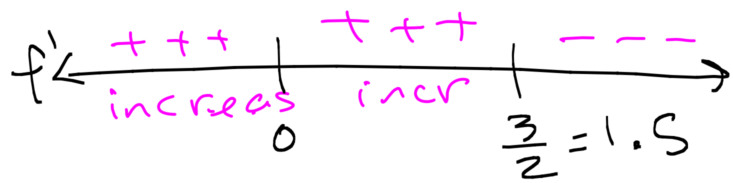
$$= -18 \quad (-)$$

Concave down at $x = \frac{3}{2}$

Relative max at $x = \frac{3}{2}$

1st derivative Test:

$f'(x) = -8x^3 + 12x^2$



$(-\infty, 0)$: Test $x = -1$

$$f'(-1) = -8(-1)^3 + 12(-1)^2$$

$$= 8 + 12$$

$$(+)$$

$\left(0, \frac{3}{2}\right)$: Test $x = 1$

$$f'(1) = -8(1)^3 + 12(1)^2$$

$$= -8 + 12$$

$$= 4 \quad (+)$$

$\left(\frac{3}{2}, \infty\right)$: Test $x = 2$

$$f'(2) = -8(2)^3 + 12(2)^2$$

$$= -64 + 48$$

$$(-)$$

$\boxed{\text{Relative max at } x = \frac{3}{2}}$