

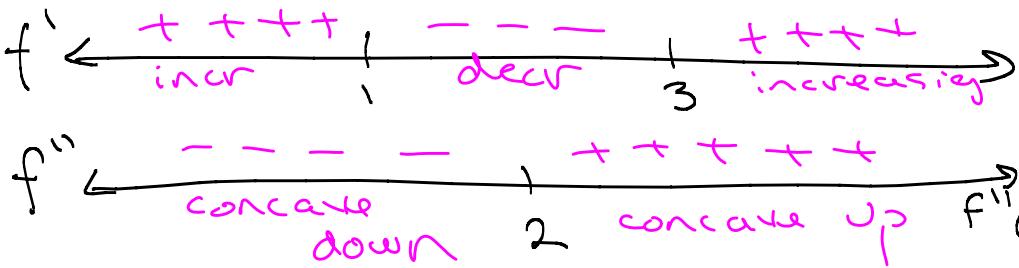
3.6: A Summary of Curve Sketching

Steps for Curve Sketching

1. Determine the domain of f .
2. Find the x -intercepts and y -intercept, if any.
3. Determine the “end behavior” of f , that is, the behavior for large values of $|x|$ (limits at infinity). *(I often skip this, ... it comes out in the incr/decras.)*
4. Find the vertical, horizontal, and oblique asymptotes, if any.
5. Determine the intervals where f is increasing/decreasing.
6. Find the relative extremes of f , if any. (You should find both the x - and y -values.)
7. Determine the intervals where f is concave up/concave down.
8. Find the inflection points, if any. (You should find both the x - and y -values.)
9. Plot more points if necessary, and sketch the graph.

Example 1: Sketch the graph of $f(x) = x^3 - 6x^2 + 9x$.

$$\begin{aligned}
 f'(x) &= 3x^2 - 12x + 9 \\
 &= 3(x^2 - 4x + 3) \\
 &= 3(x-3)(x-1) \\
 \text{Critical } \#s: & 3, 1 \\
 f''(x) &= 6x - 12 \\
 &= 6(x-2)
 \end{aligned}$$



Find x -intercept.

$$\text{Set } y=0: 0 = x^3 - 6x^2 + 9x$$

$$0 = x(x^2 - 6x + 9)$$

$$0 = x(x-3)(x-3)$$

$$x = 0, x = 3$$

Intercepts:

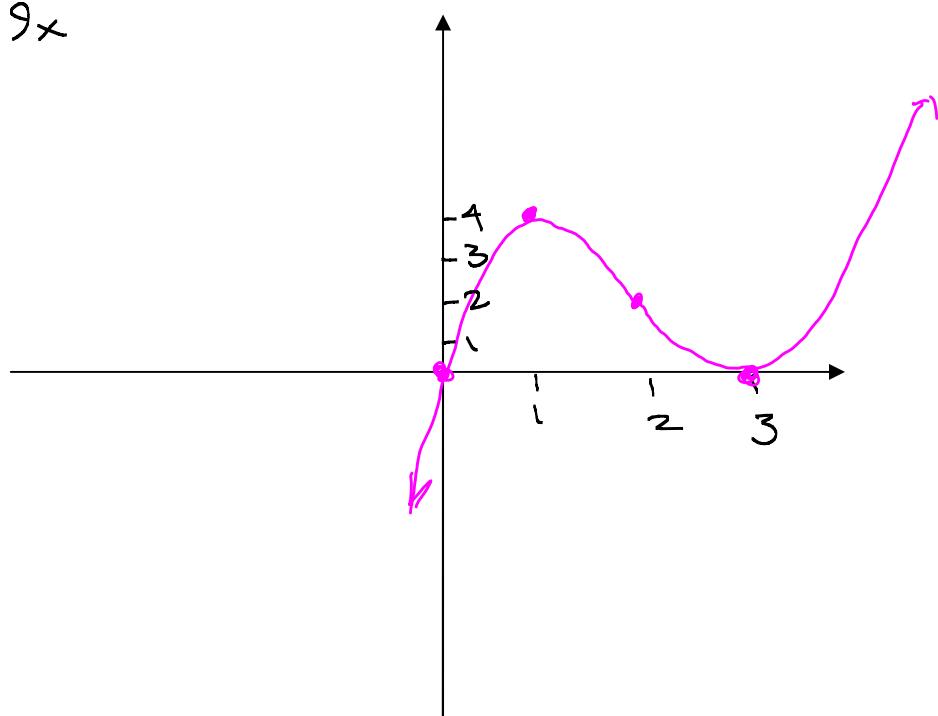
$$(0, 0), (3, 0)$$

Find y -intercept.

$$\text{Set } x=0$$

$$\begin{aligned}
 y &= 0^3 - 6(0)^2 + 9(0) \\
 &= 0
 \end{aligned}$$

$$(0, 0)$$



Calculate y-values: $f(x) = x^3 - 6x^2 + 9x$

$$f(1) = 1^3 - 6(1)^2 + 9(1) = 4$$

$$f(3) = 0$$

$$f(2) = 2^3 - 6(2)^2 + 9(2) = 2$$

Relative max: $(1, 4)$

Relative min: $(3, 0)$

Inflection Pt: $(2, 2)$

Example 2: Sketch the graph of $f(x) = 3x^4 + 4x^3$.

$$f'(x) = 12x^3 + 12x^2$$

$$= 12x^2(x+1)$$

$x=0, x=-1$ critical pts

$$f''(x) = 36x^2 + 24x$$

$$= 12x(3x+2)$$

$$x=0, x=-\frac{2}{3}$$

1st derivative

$(-\infty, -1)$: Test $x=-2$

$$f'(-2) = 12(-2)^2(-2+1) \Rightarrow (+)(-) \Rightarrow (-)$$

$(-1, 0)$: Test $x=-\frac{1}{2}$

$$f'\left(-\frac{1}{2}\right) = 12\left(-\frac{1}{2}\right)^2\left(-\frac{1}{2}+1\right) \Rightarrow (+)(+) \Rightarrow (+)$$

$(0, \infty)$: Test $x=1$

$$f'(1) = 12(1)^2(1+1) \Rightarrow (+)$$

intercepts:

$$(0, 0)$$

$$\left(-\frac{4}{3}, 0\right)$$

Relative min at $(-1, -1)$

inflection pts:

$$\left(-\frac{2}{3}, -\frac{16}{27}\right)$$

$$(0, 0)$$

Find x -intercepts:

$$\text{set } y=0: 0 = 3x^4 + 4x^3$$

$$0 = x^3(3x+4)$$

$$x=0, x=-\frac{4}{3}$$

Find y -intercept: set $x=0$

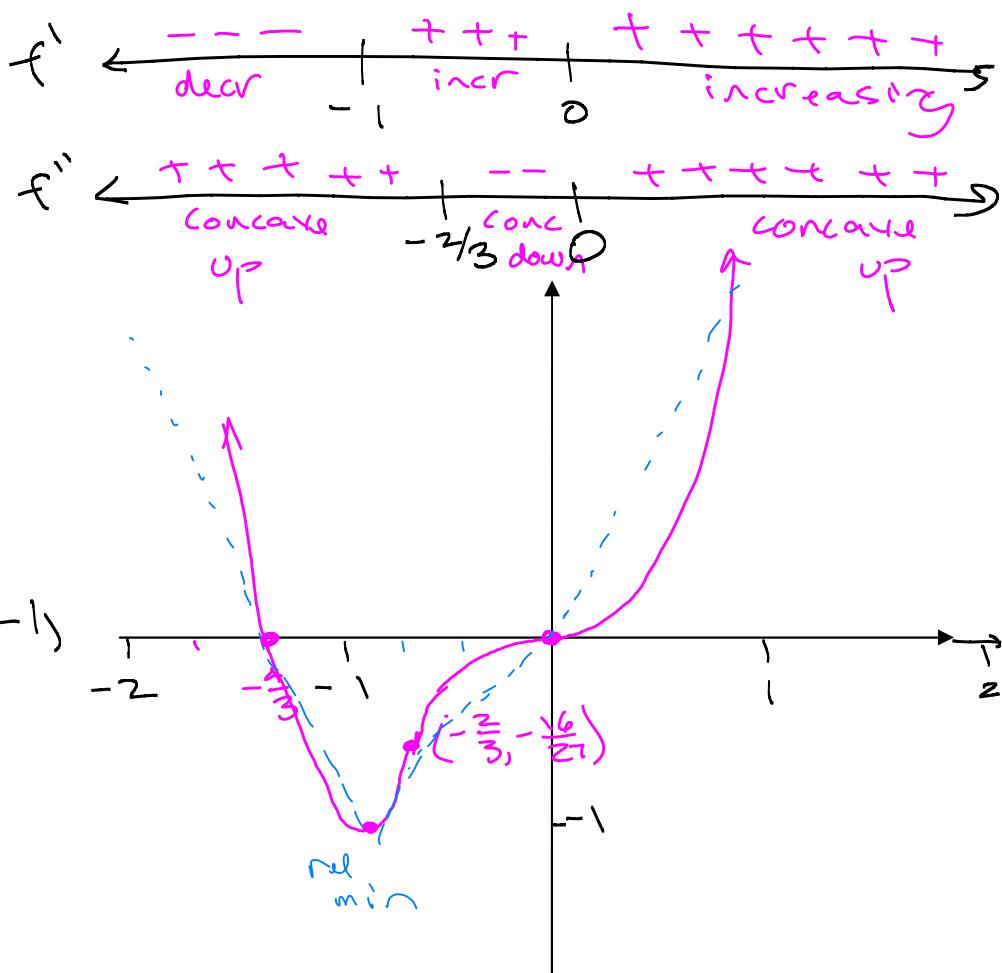
$$y = 3(0)^4 + 4(0)^3 = 0$$

Increasing on $(-1, \infty)$

Decreasing on $(-\infty, -1)$

Concave up on

Concave down on



cont'd next page

$$\text{Find } y\text{-values: } f(-1) = 3(-1)^4 + 4(-1)^3 \\ = 3 - 4 = -1$$

$$\underline{2^{\text{nd}} \text{ derivative: } f''(x) = 12x(3x+2)}$$

$$(-\infty, -\frac{2}{3}): \text{ Test } x = -1$$

$$f''(-1) = 12(-1)(3(-1)+2) \\ = -12(-3+2) \\ = -12(-1) \\ = +12 \\ (+)$$

$$(-\frac{2}{3}, 0): \text{ Test } x = -\frac{1}{3}$$

$$f''(-\frac{1}{3}) = 12(-\frac{1}{3})(3(-\frac{1}{3})+2) \\ = -4(-1+2) \\ \Rightarrow (-)(+) \\ \Rightarrow (-)$$

inflection pts at $x=0, x = -\frac{2}{3}$.

$$\underline{\text{Find } y\text{-values}} \quad f(0) = 0$$

$$f(-\frac{2}{3}) = 3(-\frac{2}{3})^4 + 4(-\frac{2}{3})^3 \\ = \frac{16}{27} + 4\left(-\frac{8}{27}\right) \\ = \frac{16}{27} - \frac{32}{27} \\ = -\frac{16}{27}$$

Example 3: Sketch the graph of $f(x) = \frac{2x}{x^2 - 1}$.

$$f'(x) = \frac{(x^2 - 1)(2) - 2x(2x)}{(x^2 - 1)^2} = \frac{2x^2 - 2 - 4x^2}{(x^2 - 1)^2} = \frac{-2x^2 - 2}{(x^2 - 1)^2}$$

$$f'(x) \Rightarrow \frac{(-)(+)}{(+)} \Rightarrow (-)$$

for all x

$$= \frac{-2(x^2 + 1)}{(x^2 - 1)^2}$$



Denominator is 0 for $x = \pm 1$
Numerator is never 0

$$f(x) = \frac{2x}{(x+1)(x-1)}$$

f is undefined at $x = 1, x = -1$ (vertical asymptotes)

f is 0 at $x = 0$.
So x -intercept = 0.

$$\text{Set } y = 0 : 0 = \frac{2x}{(x+1)(x-1)}$$

$$0 = 2x$$

$$0 = x$$

Decreasing on
 $(-\infty, -1), (-1, 1), (1, \infty)$
No relative extrema

Concave up on
 $(-1, 0)$ and $(1, \infty)$

Concave down on
 $(-\infty, -1)$ and $(0, 1)$

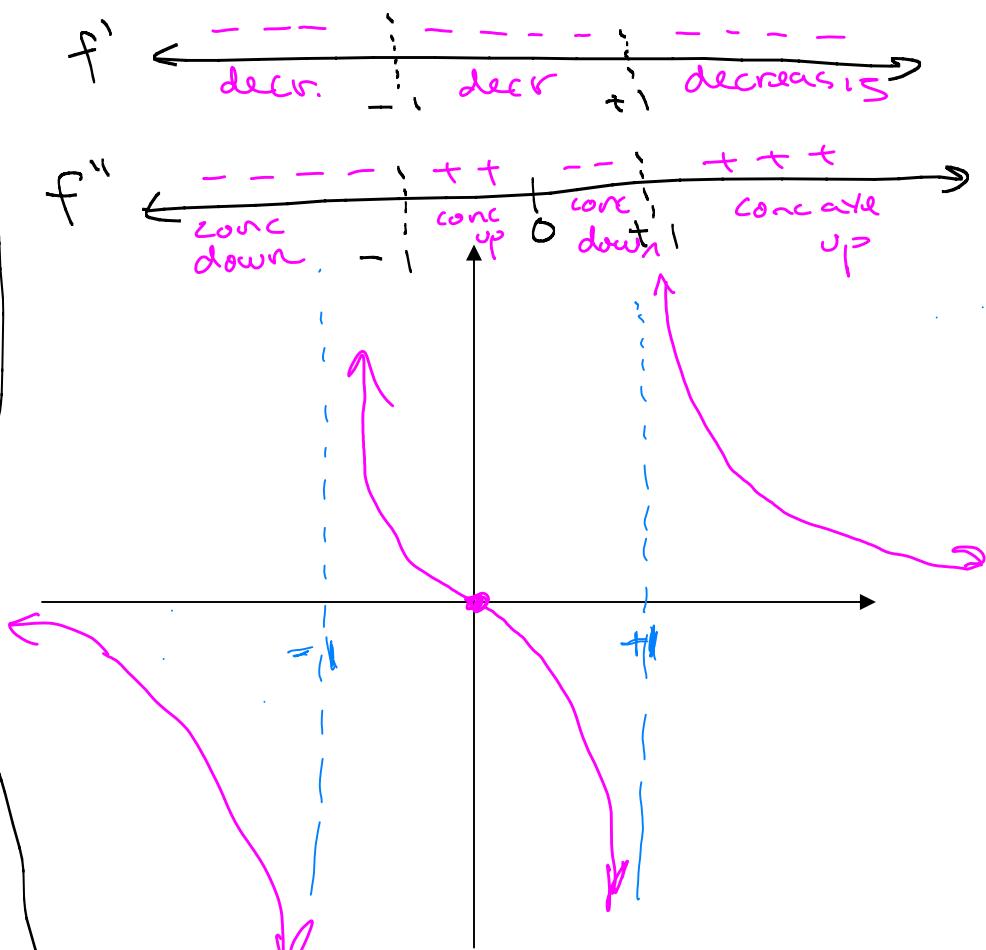
Inflection Pt:
 $(0, 0)$

x -intercept: 0
or $(0, 0)$

y -intercept: 0
(or $(0, 0)$)

Vertical Asymptotes:
 $x = \pm 1$

Horiz Asy: $y = 0$



Ex 3 cont'd

$$f'(x) = \frac{-2x^2 - 2}{(x^2 - 1)^2} = \frac{-2(x^2 + 1)}{(x^2 - 1)^2}$$

$$f''(x) = \frac{4x(x^2 + 3)}{(x^2 - 1)^3}$$

Where is $f''(x) = 0$? at $x = 0$

Where is $f''(x)$ undefined? at $x = \pm 1$, same as for original function

$$(-\infty, -1): \text{Test } x = -2$$

$$f''(x) \Rightarrow \frac{4(-2)(+)}{(-2^2 - 1)^3} \Rightarrow \frac{(-)(+)}{(+)} \Rightarrow (-)$$

$$(-1, 0): \text{Test } x = -\frac{1}{2}$$

$$f''(-\frac{1}{2}) \Rightarrow \frac{4(-\frac{1}{2})(+)}{(-\frac{1}{2})^2 - 1)^3} \Rightarrow \frac{(-)}{(-)^3} \Rightarrow \frac{(-)}{(-)} \Rightarrow (+)$$

$$(0, 1): \text{Test } x = \frac{1}{2}$$

$$f''(\frac{1}{2}) \Rightarrow \frac{(+)}{(-)^3} \Rightarrow \frac{(+)}{(-)} \Rightarrow (-)$$

$$(1, \infty): \frac{(+)}{(+)^3} \Rightarrow (+)$$

Look for horizontal asymptotes

$$y = \frac{2x}{x^2 - 1}$$

$$\text{as } x \rightarrow \pm \infty, y \rightarrow \frac{2x}{x^2} \rightarrow \frac{\frac{2}{x}}{\frac{1}{x^2}} \rightarrow 0$$

$$\lim_{x \rightarrow \pm \infty} \frac{2x}{x^2 - 1} = \lim_{x \rightarrow \pm \infty} \frac{\frac{2}{x}}{1 - \frac{1}{x^2}} = \frac{0}{1 - 0} = 0$$

Horiz. Asymptote: $y = 0$

Example 4: Sketch the graph of $f(x) = \frac{x^2+1}{x^2-4}$

$$f'(x) = \frac{-10x}{(x^2-4)^2}$$

$f'(x) = 0$ for $x=0$; undefined for $x=\pm 2$

$$f''(x) = \frac{10(3x^2+4)}{(x^2-4)^3}$$

$f''(x)$ is never 0; undefined at $x=\pm 2$

Find x-intercepts:

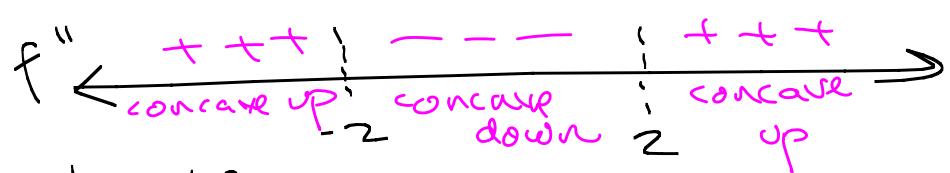
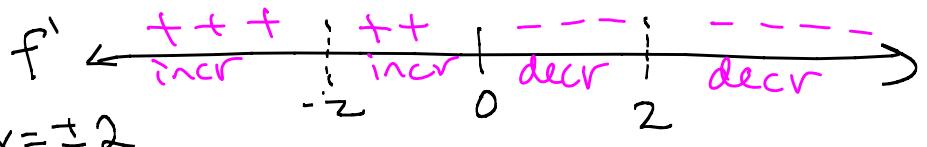
$$\text{Set } y=0: 0 = \frac{x^2+1}{x^2-4}$$

Numerator is never 0.

No x-intercepts

Vertical Asymptotes:

$$x = \pm 2$$



Find y-intercept:

$$\text{Set } x=0: y = \frac{0^2+1}{0^2-4} = -\frac{1}{4}$$

y-intercept: $-\frac{1}{4}$ or $(0, -\frac{1}{4})$

Find horizontal asymptote

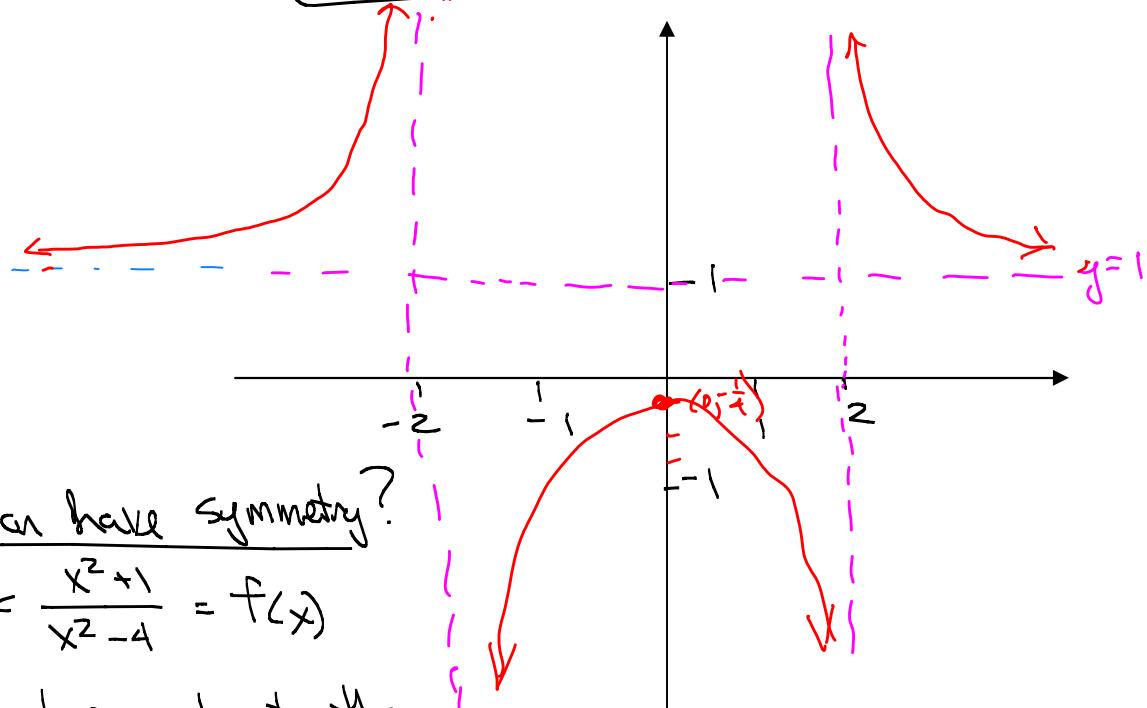
$$\text{As } x = \pm \infty, y = \frac{x^2+1}{x^2-4} \rightarrow \frac{x^2+1}{x^2} = 1$$

Horizontal asymptote: $y = 1$

Relative Max.:

$$f(0) = -\frac{1}{4}$$

$$(0, -\frac{1}{4})$$



Inflection Points:

None
Does this function have symmetry?

$$f(-x) = \frac{(-x)^2+1}{(-x)^2-4} = \frac{x^2+1}{x^2-4} = f(x)$$

So it is symmetric about the y-axis.
(It's an even function)

Example 5: Sketch the graph of $f(x) = \frac{x^2 - 4}{x + 3}$. $= \frac{(x+2)(x-2)}{x+3}$

x-intercept: $-2, 2$

y-intercept: $-\frac{4}{3}$

Set $x=0$: $f(0) = \frac{0^2 - 4}{0 + 3} = -\frac{4}{3}$

Find slant asymptote:

$$\begin{array}{r} x - 3 \\ x+3 \overline{)x^2 + 0x - 4} \\ - (x^2 + 3x) \\ \hline -3x - 4 \\ - (-3x - 9) \\ \hline 5 \end{array}$$

$$f'(x) = \frac{x^2 + 6x + 4}{(x+3)^2}$$

Numerator doesn't factor.
Use quadratic formula. $x^2 + 6x + 4 = 0$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-6 \pm \sqrt{20}}{2}$$

$$x = \frac{-6 \pm 2\sqrt{5}}{2}$$

$$x = -3 \pm \sqrt{5} \approx -0.763, -5.236$$

(critical numbers)

$$f''(x) = \frac{10}{(x+3)^3}$$

Find y-values for critical numbers:
 $f(-3 + \sqrt{5}) \approx -1.53$

Vertical asymptote: $x = -3$

Look for horizontal Asymptote:

As $x \rightarrow \pm\infty$, $y \rightarrow \frac{x^2}{x} = x \rightarrow \pm\infty$

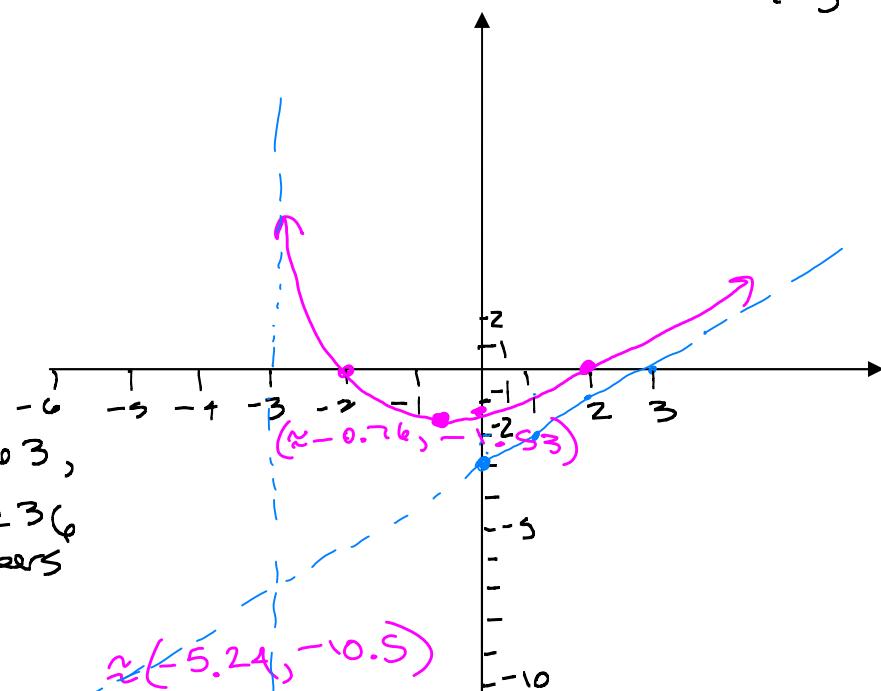
It has a slant asymptote

$$y = \frac{x^2 - 4}{x+3} = x - 3 + \frac{5}{x+3}$$

Slant asymptote: $y = x - 3$

$$f' \leftarrow \begin{array}{c} +++ \\ \text{incr} \end{array} \mid \begin{array}{c} -- \\ \text{decr} \end{array} \mid \begin{array}{c} --- \\ \text{decr} \end{array} \mid \begin{array}{c} ++ \\ \text{incr} \end{array} \\ \approx -5.2 \quad -3 \quad \approx -0.76 \end{math>$$

$$f'' \leftarrow \begin{array}{c} --- \\ \text{conc. down} \end{array} \mid \begin{array}{c} ++ \\ \text{conc. up} \end{array} \\ -3 \end{math>$$



$$f(-3 - \sqrt{5}) \approx -10.5$$

$$f(x) = 5x^{\frac{2}{3}} - x^{\frac{5}{3}} = 5\sqrt[3]{x^2} - \sqrt[3]{x^5}$$

3.6.6

Example 6: Sketch the graph of $f(x) = 5x^{\frac{2}{3}} - x^{\frac{5}{3}}$.

Find x-intercepts: Set $y=0$: $0 = 5x^{\frac{2}{3}} - x^{\frac{5}{3}}$

$$0 = x^{\frac{2}{3}}(5 - x^{\frac{3}{2}}) = x^{\frac{2}{3}}(5 - x)$$

$$0 = x^{\frac{2}{3}} \quad 5 - x = 0$$

$$x = 0 \quad 5 = x$$

x-intercepts: 0, 5

Find y-intercept:

Set $x=0$: $y = 5(0)^{\frac{2}{3}} - 0^{\frac{5}{3}} = 0$

y-intercept: 0

$$f(x) = 5\sqrt[3]{x^2} - \sqrt[3]{x^5}$$

Find y-values for critical #s
and inflection points

(work on next page would
be done 1st)

Critical #s: 0, 2

$$f(0) = 0$$

$$f(2) = 5\sqrt[3]{2^2} - \sqrt[3]{2^5}$$

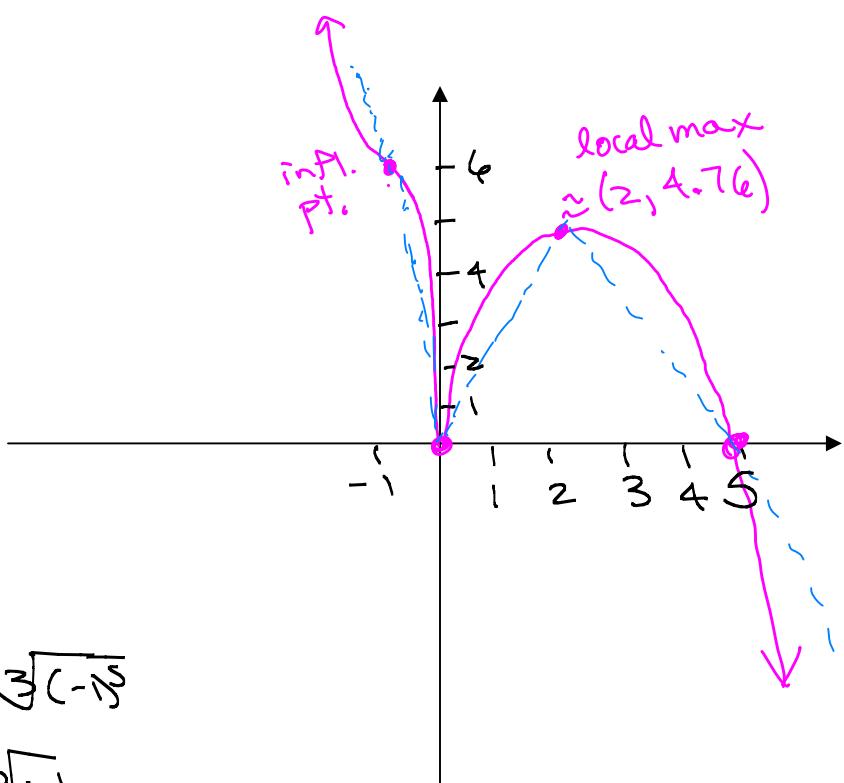
$$\approx 4.7$$

Inflection pt at $x = -1$:

$$f(-1) = 5\sqrt[3]{(-1)^2} - \sqrt[3]{(-1)^5}$$

$$= 5\sqrt[3]{1} - \sqrt[3]{-1}$$

$$= 5(1) - (-1) = 5 + 1 = 6$$



Example 6 cont'd:

$$f(x) = 5x^{\frac{2}{3}} - x^{\frac{5}{3}}$$

$$f'(x) = 5\left(\frac{2}{3}\right)x^{-\frac{1}{3}} - \frac{5}{3}x^{\frac{2}{3}}$$

$$= \frac{10}{3\sqrt[3]{x}} - \frac{5\sqrt[3]{x^2}}{3} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}}$$

$$= \frac{10}{3\sqrt[3]{x}} - \frac{5\sqrt[3]{x^3}}{3\sqrt[3]{x}} = \frac{10 - 5x}{3\sqrt[3]{x}}$$

Find critical numbers: $f'(x) = 0$ when $10 - 5x = 0$
 $10 = 5x$
 $2 = x$

$f'(x)$ is undefined for $x = 0$

Critical numbers: 2 and 0

$$f'(x) = \frac{10 - 5x}{3x^{\frac{1}{3}}} = \frac{10}{3}x^{-\frac{1}{3}} - \frac{5}{3}x^{\frac{2}{3}}$$

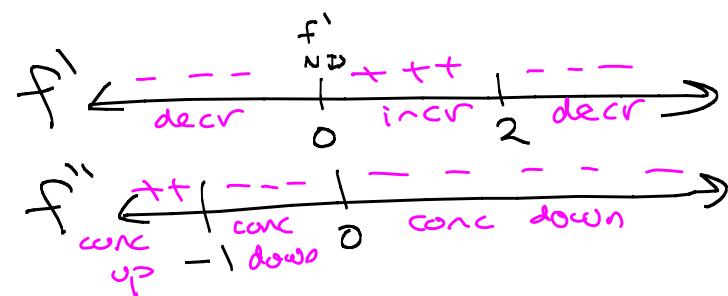
$$f''(x) = \frac{10}{3}\left(-\frac{1}{3}\right)x^{-\frac{4}{3}} - \frac{5}{3}\left(\frac{2}{3}\right)x^{-\frac{1}{3}}$$

$$= -\frac{10}{9x^{\frac{4}{3}}} - \frac{10}{9x^{\frac{1}{3}}} \cdot \frac{x}{x}$$

$$= \frac{-10 - 10x}{9x^{\frac{4}{3}}} = \frac{-10(x+1)}{9x^{\frac{4}{3}}} = \frac{-10(x+1)}{9\sqrt[3]{x^4}}$$

$$f''(x) = 0 \text{ for } x = -1$$

$$f''(x)$$
 is undefined for $x = 0$



We did not do this one in class but I wrote up the solution for you.

3.6.7

Example 7: Sketch the graph of $f(x) = x + \cos x$ on the interval $[-2\pi, 2\pi]$.

Find x -intercepts: Set $y=0$: $0 = x + \cos x$

$$-x = \cos x$$

can't solve
algebraically

Find y -intercept: Set $x=0$

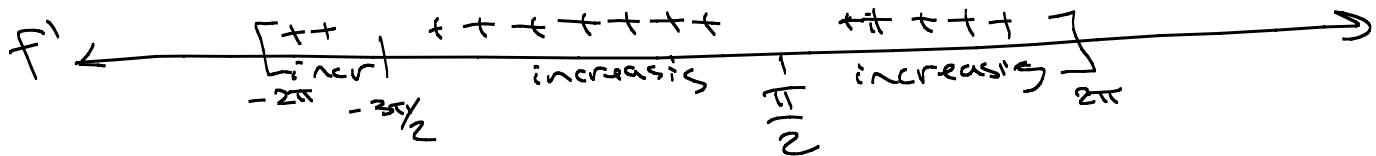
$$y = 0 + \cos 0 = 0 + 1 = 1$$

$$f'(x) = 1 - \sin x$$

$$\text{Set } f'(x)=0: 0 = 1 - \sin x$$

$$\sin x = 1$$

$$x = \frac{\pi}{2}, -\frac{3\pi}{2} \text{ critical HS.}$$



$$(-2\pi, -\frac{3\pi}{2}): \text{Test } x = -\frac{11\pi}{6} \Rightarrow f'(-\frac{11\pi}{6}) = 1 - \sin(-\frac{11\pi}{6}) = 1 - \frac{1}{2} = \frac{1}{2} (+)$$

$$(-\frac{3\pi}{2}, \frac{\pi}{2}): \text{Test } x = 0 \Rightarrow 1 - \sin 0 = 1 - 0 = 1 (+)$$

$$(\frac{\pi}{2}, 2\pi): \text{Test } x = \pi \Rightarrow 1 - \sin \pi = 1 - 0 = 1 (+)$$

f is increasing on $(-2\pi, 2\pi)$

Find y -value at endpoints:

$$f(2\pi) = 2\pi + \cos 2\pi$$

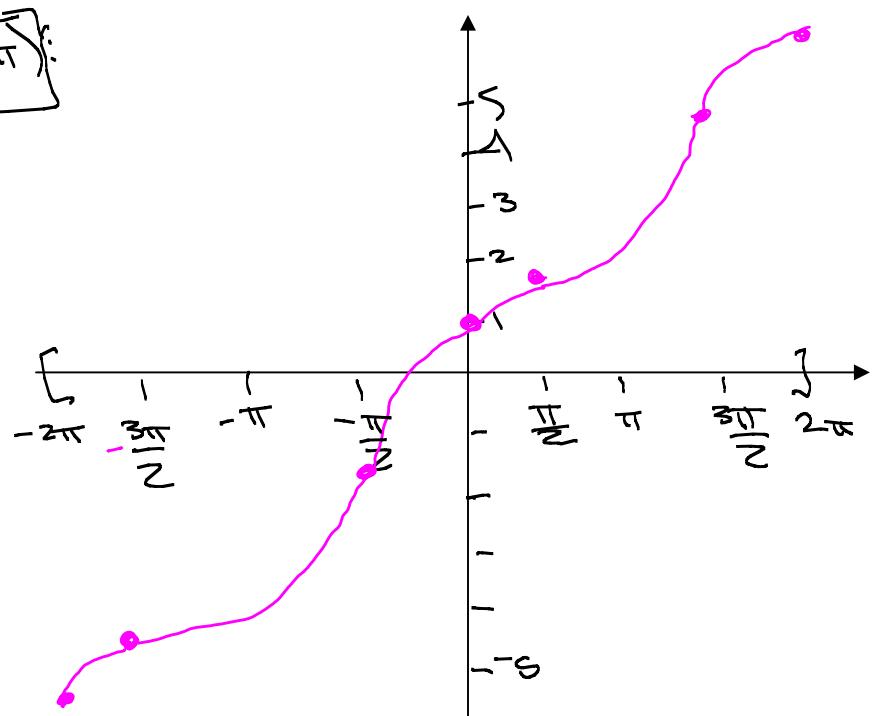
$$= 2\pi + 1$$

$$\approx 6.28 + 1$$

$$\approx 7.28$$

$$f(-2\pi) = -2\pi + \cos(-2\pi)$$

$$\approx -5.28$$



Ex. 7 cont'd:

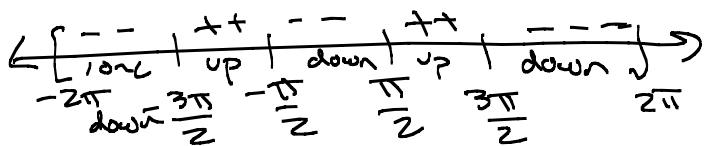
2nd derivative:

$$f'(x) = -\sin x$$

$$f''(x) = 0 - \cos x \\ = -\cos x$$

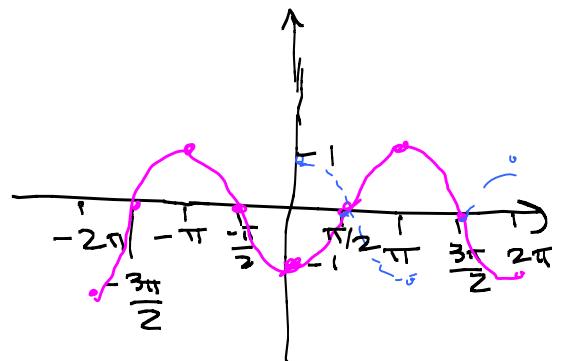
$$\text{Set } f''(x) = 0 : 0 = -\cos x \\ 0 = \cos x$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2}$$



$$f''(x) = -\cos x$$

$$\text{Graph } y = -\cos x$$



Find y-values:

$$f(-\frac{3\pi}{2}) = -\frac{3\pi}{2} + \cos(-\frac{3\pi}{2}) = -\frac{3\pi}{2} + 0 = -\frac{3\pi}{2} \approx -4.7$$

$$f(-\frac{\pi}{2}) = -\frac{\pi}{2} + \cos(-\frac{\pi}{2}) = -\frac{\pi}{2} \approx -1.57$$

$$f(\frac{\pi}{2}) = \frac{\pi}{2} + \cos(\frac{\pi}{2}) = \frac{\pi}{2} + 0 \approx 1.57$$

$$f(\frac{3\pi}{2}) \approx 4.7$$

No Relative extrema

Inflection Points at $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

Homework Questions

3.3 #45

$$f(x) = \cos^2 2x \\ = (\cos(2x))^2$$

$$f'(x) = 2(\cos(2x)) \frac{d}{dx} (\cos(2x)) \\ = 2\cos(2x)(-\sin(2x))(2) \\ = -4\cos(2x)\sin(2x)$$

Set $f'(x) = 0$: $0 = -4\cos(2x)\sin(2x)$

$$-4 = 0 \quad \text{or}$$

never true

$$\cos(2x) = 0 \\ \theta = 2x \\ \Rightarrow \cos\theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\theta = \frac{\pi}{2} + k\pi$$

$$2x = \frac{\pi}{2} + k\pi$$

$$x = \frac{\pi}{4} + \frac{k\pi}{2}$$

$$\text{or } \sin(2x) = 0$$

$$\theta = 2x \Rightarrow \sin\theta = 0$$

$$\theta = 0, \pi$$

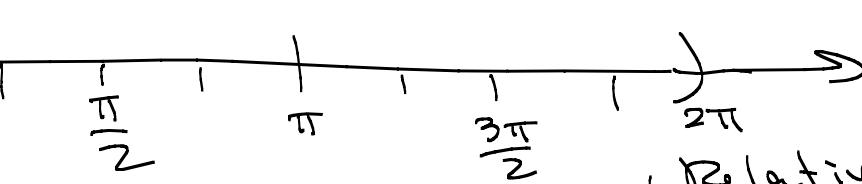
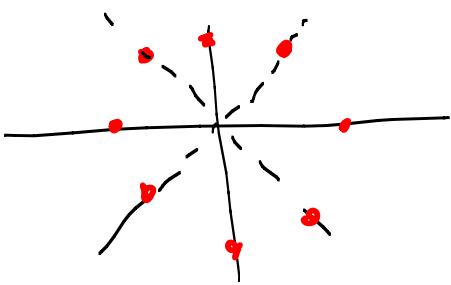
$$\theta = 0 + k\pi$$

$$2x = k\pi$$

$$x = \frac{k\pi}{2}$$

k any integer

$$k \in \mathbb{Z}$$



3.3 #29

$$f(x) = (x+2)^{2/3} = \sqrt[3]{(x+2)^2}$$

$$f'(x) = \frac{2}{3}(x+2)^{-1/3}(1) = \frac{2}{3\sqrt[3]{x+2}}$$

Relative min

$$f(2) = \sqrt[3]{(-2+2)^2} \\ = \sqrt[3]{0^2} = 0$$

where is $f'(x) = 0$? Never

Where is $f'(x)$ undefined? at $x = -2$ Only critical #

