3.9: Differentials

Differentials:



Let $d x=\Delta x$

If $y=f(x)$ is a differentiable function, we can let $d x$ represent an amount of change in $x$.

$$
\text { Then the differential } d y \text { is defined to be } d y=f^{\prime}(x) d x . \Longrightarrow \frac{d y}{d x}=f^{\prime}(x)
$$

$d y$ is an approximation to $\Delta y=f(x+\Delta x)-f(x)$, which is the actual change in $y$.
Example 1: Find the differential $d y$ for $y=\sqrt{6+x}$. Evaluate it when $x=10$ and $d x=-0.3$.
Compare it to the exact value of $\Delta y$.

$$
\begin{aligned}
y & =\sqrt{6+x} \\
& =(6+x)^{-1 / 2} \\
\frac{d y}{d x} & =\frac{1}{2}(6+x)^{-1 / 2}(1) \\
\frac{d y}{d x} & =\frac{1}{2 \sqrt{6+x}}
\end{aligned}
$$

$$
\begin{aligned}
& d y=\frac{-0.3}{2 \sqrt{6+10}} \\
& x=10 \\
& d x=-0.3
\end{aligned}
$$

Example 2: Compute $d y$ and $\Delta y$ if $y=5 x+x^{3}$ as $x$ changes from 3 to 3.05.
Compute actual change in y:

$$
\begin{aligned}
\frac{d y}{d x} & =5+3 x^{2} \rightarrow d y=\left(5+3 x^{2}\right) d x \\
d y & =\left(5+3(3)^{2}\right)(0.05) \\
x & =3 \\
d x=0.05 & =32(0.05)=1.6 \\
\Delta y & =f(3.05)-f(3) \\
& =\left[5(3.05)+(3.05)^{3}\right]-\left[5(3)+3^{3}\right] \\
& =43.622625-42 \\
& =1.622625
\end{aligned}
$$

$$
\begin{aligned}
x+d x & =10-0.3=9.7 \\
\Delta y & =f(x+d x)-f(x) \\
& =f(9.7)-f(10) \\
& =\sqrt{6+9.7}-\sqrt{6+10} \\
& =\sqrt{15.7}-\sqrt{16}=\sqrt{15.7}-4 \\
& \approx-0.037677
\end{aligned}
$$

The ingratiation of fanacion: (using a tangent line to appomimenty the function near a


$$
\begin{aligned}
& x+1 \\
& x+
\end{aligned}
$$

$$
x+\Delta x
$$

$$
\begin{aligned}
& \text { slope or } \\
& \text { line }=\frac{f(x)-f(a)}{x-a}
\end{aligned}
$$

The linearization of $f$ at $a$ is

$$
\begin{aligned}
L(x) & =f(a)+f^{\prime}(a)(x-a) \\
y & =y_{1}+m(x-x
\end{aligned}
$$

The approximation $f(x) \approx L(x)$ is the standard linear approximation of $f$ at $a$.
$\gamma$ equation of tangent line.
Example 3: Find the linearization of $f(x)=x^{4}$ at 3 . Use this to approximate (3.013) ${ }^{4}$ and

$$
\left.y\right|_{x=3.013}=\frac{108(3.03)-243}{}=82.404 \quad \begin{aligned}
& f^{\prime}(x)=4 x^{3} \\
& \text { slope of tangent sine: } m=f^{\prime}(3)=4(3)^{3}=108 \\
& \text { Fin the } y \text {-value: } f(3)=3^{4}=81 .
\end{aligned}
$$ target

line
definition of derivative at $(a, f(a))$

$$
f^{\prime}(a)=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}
$$

 particular $x$-value.)
slope of tangent ling

$$
=f^{\prime}(x)
$$



$$
\begin{aligned}
& f^{\prime}(a) \approx \frac{f(x)-f(a)}{x-a} \text { multiply both sides } \\
& f^{\prime}(a) \stackrel{\approx}{\approx} \frac{a t(x)-f(a)}{x-a} \text { multiply both sides }
\end{aligned}
$$

Example 4: Find the linearization of $f(x)=\sin x$ at $60^{\circ}$. Use this to approximate $\sin 62^{\circ}$ and $\sin 58^{\circ}$.

$$
f^{\prime}(x)=\cos x
$$

At $x=60^{\circ}$, slope $=f^{\prime}\left(60^{\circ}\right)=\cos 60^{\circ}=\frac{1}{2}$
Find point on tangent line: $f\left(60^{\circ}\right)=\sin 40^{\circ}=\frac{\sqrt{3}}{2}$

$$
\begin{aligned}
& y-y_{1}=m\left(x-x_{1}\right) \\
& y-\frac{\sqrt{3}}{2}=\frac{1}{2}\left(x-60^{\circ}\right) . \\
& \text { Linear zation: } L(x)=\frac{1}{2}\left(x-60^{\circ}\right)+\frac{\sqrt{3}}{2} \text {, where } x \text { is }
\end{aligned}
$$

$$
L\left(62^{\circ}\right)=\frac{1}{2}\left(62^{\circ}-60^{\circ}\right)+\frac{\sqrt{3}}{2}
$$

$$
=\frac{2^{0}}{2}+\frac{\sqrt{3}}{2}
$$

$=\frac{2^{0}}{2}\left(\frac{\frac{2}{11}}{180^{\circ}}\right)+\frac{\sqrt{3}}{2} \quad \begin{gathered}\text { Need to change it to a real \#, } \\ \text { otherwise it does not male sense }\end{gathered}$

$$
=\frac{\pi}{180}+\frac{\sqrt{3}}{2} \approx 0.88348
$$ to add it to $\frac{\sqrt{3}}{2}$

Compare to calculating $\sin 62^{\circ}$ on calculator: $\sin 62^{\circ} \approx 0.0829476$

$$
L\left(50^{\circ}\right)=\frac{1}{2}\left(58^{\circ}-60^{\circ}\right)+\frac{\sqrt{3}}{2}=-\frac{2}{2}\left(\frac{\pi}{180^{\circ}}\right)+\frac{\sqrt{3}}{2}=-\frac{\pi}{180}+\frac{\sqrt{3}}{2} \approx 0.848572
$$

Error propagation: Calculatry $\sin 58^{\circ}$ on calculator results in 0.84804809
If $x$ is the measured value of a variable and $x+\Delta x$ is the exact value of the variable, then $\Delta x$ is the measurement error. If we use the measured value of $x$ to calculate the value of a function $f$, then the propagated error is $\Delta y=f(x+\Delta x)-f(x)$. The propagated error can be estimated by calculating $d y=f^{\prime}(x) d x \approx f^{\prime}(x) \Delta x$.

Estimating propagated error:
If $x$ is the measured value of a variable and $\Delta x$ is the measurement error, then:
Estimated propagated error: $d y=f^{\prime}(x) d x \approx f^{\prime}(x) \Delta x$
Estimated relative error: $\frac{d y}{y}$

Example 5: The measurement of the radius of a circle is 20 inches with a maximum error of 0.10 inch. Approximate the maximum propagated error and the relative error in computing the area and the circumference of the circle. Circumference $\rightarrow$ Prop error: $d C)=2 \pi(0.10)$
Area

$$
\begin{aligned}
& \begin{array}{l}
A=\pi r^{2} \\
d A
\end{array} \left\lvert\, \begin{array}{l}
C=2 \pi r \\
\frac{d c}{d r}=2 \pi
\end{array}\right. \\
& \begin{array}{l}
C=2 \pi r \\
\frac{d c}{d r}=2 \pi \\
d c=2 \pi d r
\end{array} \\
& \begin{array}{l}
r=20=0.2 \pi \\
d r=10
\end{array} \\
& \approx 0.628 \mathrm{in} \\
& \text { Relative ensor: } \frac{d C}{C}=\frac{0.2 \pi}{2 \pi r}=\frac{2 \pi}{20 \pi r}
\end{aligned}
$$

$$
\begin{aligned}
& d A=2 \pi r d r \\
& \text { Propagated error }=\left.d A\right|_{\left\lvert\, \begin{array}{l}
r=20 \\
d r \\
d r 0.10
\end{array}\right.}
\end{aligned}
$$

Example 6: The measurements of the height and inside radius of a right cylinder are 50 feet and 30 feet, respectively. The maximum possible error in each measurement is about 3 inches per each 10 feet of measured length. Approximate the maximum propagated error and the relative error in computing the volume of the cylinder.
$V=\pi_{r}^{2} h \quad W_{e} l l$ need the product rub, because we have both $h$ and $r$.

$$
\frac{d V}{d r}=\pi r^{2} \frac{d h}{d r}+h(2 \pi r)
$$

multiply both sides by dr:

$$
d V=\pi r^{2} d h+2 \pi r h d r
$$

For $h$, measurement error $d h$ is $5(3)=15 \sin =1 \sin \left(\frac{1 \mathrm{ft}}{12 \mathrm{in}}\right)=1.25 \mathrm{ft}$
For $r$, measurement error dr is $3(3)=9 \mathrm{in}=\operatorname{lin}\left(\frac{17}{12 \mathrm{in}}\right)=0.75 \mathrm{ft}$

$$
\begin{aligned}
& \left.d V\right|_{r=30 f t}=\pi(30 \mathrm{ft})^{2}(1.25 \mathrm{ft})+2 \pi(30 \mathrm{ft})(50 \mathrm{ft})(0.75 \mathrm{ft}) \\
& \begin{array}{l}
r=30 \mathrm{ft} \\
h=50 \mathrm{ft}
\end{array} \quad=11250 \mathrm{ft}^{3}+22500 \mathrm{ft}^{3}=3375 \pi \mathrm{ft}^{3} \\
& d r=0.75 \mathrm{ft} \\
& d \mathrm{~h}=1.2 \mathrm{sft} \\
& V=\left.\pi r^{2} h \Rightarrow V\right|_{\begin{array}{l}
r=30 \mathrm{ft} \\
h=50 \mathrm{ft}
\end{array}}=\pi(30 \mathrm{ft})^{2}(50 \mathrm{ft})=45000 \pi \mathrm{ft}^{3} \\
& \text { Relative error }=\frac{d V}{V}=\frac{3375 \pi f+3}{45000 \pi f+3}=\frac{3375}{45000}=0.075 \text {. } \\
& \Rightarrow 7.5 \frac{0}{0}
\end{aligned}
$$

