

4.1: Antiderivatives and Indefinite Integration

Definition: An *antiderivative* of f is a function whose derivative is f .

i.e. A function F is an antiderivative of f if $F'(x) = f(x)$.

Example 1: $F(x) = x^3 + 5x$ is an antiderivative of $f(x) = 3x^2 + 5$.

$$\text{Check: } F'(x) = 3x^2 + 5 \quad \checkmark$$

What are some more antiderivatives of $f(x) = 3x^2 + 5$?

$$G(x) = x^3 + 5x + 12$$

$$H(x) = x^3 + 5x - \frac{17\pi}{2}$$

All antiderivatives take the form $x^3 + 5x + C$

So we have a whole “family” of antiderivatives of f .

where C is
a constant

Definition: A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I .

Theorem: If F is an antiderivative of f on an interval I , then all antiderivatives of f on I will be of the form

$$F(x) + C$$

where C is an arbitrary constant.

Example 2: Find the general form of the antiderivatives of $f(x) = 3x^2 + 5$.

$$F(x) = x^3 + 5x + C$$

Example 3: Find the general form of the antiderivatives of $f(x) = 6x^5 + \cos x$.

$$F(x) = x^6 + \sin x + C$$

$$\text{Check: } F'(x) = 6x^5 + \cos x \quad \checkmark$$

Integration:

Integration is the process of finding antiderivatives.

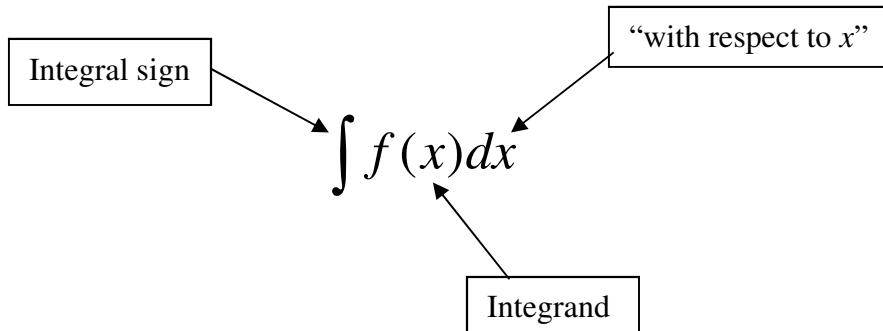
$$\int f(x) dx$$

$\int f(x) dx$ is called the *indefinite integral* of f .

$\int f(x) dx$ is the family of antiderivatives, or the most general antiderivative of f .

This means: $\int f(x) dx = F(x) + c$, where $F'(x) = f(x)$.

The c is called the *constant of integration*.



Example 4: Find $\int (3x^2 + 5) dx$.

$$\int (3x^2 + 5) dx = \boxed{x^3 + 5x + c}$$

Example 5: Find $\int 6x^5 + \cos x dx$.

$$\int (6x^5 + \cos x) dx = \boxed{x^6 + \sin x + c}$$

Example 6: Find $\int \sec^2 dx$.

$$\int \sec^2 x dx = \boxed{\tan x + c}$$

Check: $\frac{d}{dx} (\tan x + c) = \sec^2 x$

Rules for Finding Antiderivatives:

Notation in this table: F is an antiderivative of f , G is an antiderivative of g ,

Function	Antiderivative
k	$kx + c$
$kf(x)$	$kF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
x^n for $n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

$$1. \int k \, dx = kx + c \quad (k \text{ a constant})$$

Power Rule \rightarrow

$$2. \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\xrightarrow{\text{Ex.}} \int x^4 \, dx = \frac{x^5}{5} + c$$

$$3. \int k f(x) \, dx = k \int f(x) \, dx \quad (k \text{ a constant})$$

Check:

$$\frac{d}{dx} \left(\frac{x^5}{5} \right) = \frac{1}{5} x^4$$

$$4. \int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx$$

$$= \frac{1}{5} (\int x^4) = x^4 \quad \checkmark$$

$$5. \int \cos x \, dx = \sin x + c$$

$$6. \int \sin x \, dx = -\cos x + c$$

$$7. \int \sec^2 x \, dx = \tan x + c$$

$$8. \int \sec x \tan x \, dx = \sec x + c$$

Example 7: Find the general antiderivative of $f(x) = \frac{1}{2}$.

$$F(x) = \frac{1}{2}x + c$$

$$\text{Check: } \frac{d}{dx} \left(\frac{1}{2}x \right) = \frac{1}{2} \quad \checkmark$$

$$\frac{d}{dx} (\cos x) = -\sin x$$

↳ $\frac{d}{dx} (-\cos x) = \sin x \Rightarrow \int \sin x \, dx = -\cos x + C$

$$\frac{d}{dx} (\sin x) = \cos x \Rightarrow \int \cos x \, dx = \sin x + C$$

$$\frac{d}{dx} (\cot x) = -\csc^2 x$$

↳ $\frac{d}{dx} (-\cot x) = \csc^2 x \Rightarrow \int \csc^2 x \, dx = -\cot x + C$

$$\frac{d}{dx} (\csc x) = -\csc x \cot x \Rightarrow \int \csc x \cot x \, dx = -\csc x + C$$

Example 8: Find $\int x^3 dx$.

$$\int x^3 dx = \frac{x^{3+1}}{3+1} + C = \boxed{\frac{x^4}{4} + C}$$

Example 9: Find $\int 7x^2 dx$.

$$\int 7x^2 dx = 7 \int x^2 dx = 7 \cdot \frac{x^3}{3} + C = \boxed{\frac{7}{3}x^3 + C}$$

Check: $\frac{d}{dx} \left(\frac{7}{3}x^3 \right) = \frac{7}{3}(3x^2) = 7x^2 \checkmark$

Example 10: Find $\int \frac{1}{x^5} dx$.

$$\int \frac{1}{x^5} dx = \int x^{-5} dx = \frac{x^{-5+1}}{-5+1} + C = \frac{x^{-4}}{-4} + C = \boxed{-\frac{1}{4x^4} + C}$$

Check: $\frac{d}{dx} \left(-\frac{1}{4}x^{-4} \right) = -\frac{1}{4}(-4x^{-5}) = x^{-5} \checkmark$

Example 11: Find the general antiderivative of $f(x) = \frac{5}{x^2}$.

Ex [10.5] $\int x^{12} dx = \frac{x^{13}}{13} + C$

$$f(x) = 5x^{-2}$$

$$F(x) = \int 5x^{-2} dx = 5 \frac{x^{-2+1}}{-2+1} + C$$

$$= \frac{5x^{-1}}{-1} + C = -\frac{5}{x} + C$$

antiderivative is $F(x) = -\frac{5}{x} + C$

Example 12: $\int (6x^2 - 3x + 9) dx$

$$\int 6x^2 dx - \int 3x dx + \int 9 dx$$

$$= 6 \int x^2 dx - 3 \int x dx + 9 \int dx$$

$$= 6 \cdot \frac{x^3}{3} + C_1 - 3 \cdot \frac{x^2}{2} + C_2 + 9x + C_3$$

$$= \boxed{2x^3 - \frac{3x^2}{2} + 9x + C}$$

Example 13: Find $\int 3\sqrt{x} dx$.

$$3 \int x^{1/2} dx = 3 \cdot \frac{x^{3/2}}{3/2} + C$$

$$= 3 \cdot \frac{2}{3} x^{3/2} + C$$

$$= \boxed{2x^{3/2} + C}$$

$$C = C_1 + C_2 + C_3$$

Check: $\frac{d}{dx} \left(2x^3 - \frac{3}{2}x^2 + 9x \right)$

$$= 6x^2 - \frac{3}{2} \cdot 2x + 9 \checkmark$$

Check: $\frac{d}{dx} (2x^{3/2})$

$$= 2 \cdot \frac{3}{2} x^{1/2} = 3x^{1/2} \checkmark$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

.4.1.5

Example 14: Find $\int (3\cos x + 5\sin x) \, dx$.

$$3 \int \cos x \, dx + 5 \int \sin x \, dx = 3\sin x + 5(-\cos x) + C$$

$$= 3\sin x - 5\cos x + C$$

Check: $\frac{d}{dx} (3\sin x - 5\cos x)$

$$= 3\cos x - 5(-\sin x) = 3\cos x + 5\sin x \checkmark$$

Example 15: Find $\int \frac{x^7 - \sqrt[3]{x} + 3x^2}{x^5} \, dx$.

$$\int \left(\frac{x^7}{x^5} - \frac{x^{1/3}}{x^5} + \frac{3x^2}{x^5} \right) dx = \int \left(x^2 - x^{-14/3} + 3x^{-3} \right) dx$$

$$= \frac{x^3}{3} - \frac{x^{-11/3}}{-1/3} + \frac{3x^{-2}}{-2} + C$$

Example 16: Find the general antiderivative of $f(\theta) = \frac{\sin \theta}{3}$.

$$= \boxed{\frac{x^3}{3} + \frac{3}{11}x^{-1/3} - \frac{3}{2}x^{-2} + C}$$

$$F(\theta) = \int \frac{1}{3} \sin \theta \, d\theta$$

$$= \frac{1}{3} (-\cos \theta) + C = -\frac{1}{3} \cos \theta + C$$

$$F(\theta) = -\frac{1}{3} \cos \theta + C$$

Example 17: $\int \left(\sqrt[3]{x} + \frac{2}{\sqrt{x}} \right) dx$

$$\int (x^{1/3} + 2x^{-1/2}) dx = \frac{x^{1/3+1}}{1/3+1} + \frac{2x^{-1/2+1}}{-1/2+1} + C = \frac{x^{4/3}}{4/3} + \frac{2x^{1/2}}{1/2} + C$$

$$= \frac{3}{4}x^{4/3} + 2x^{1/2} + C$$

$$= \boxed{\frac{3}{4}x^{4/3} + 4x^{1/2} + C}$$

Check: $\frac{d}{dx} \left(\frac{3}{4}x^{4/3} + 4x^{1/2} \right) = \frac{3}{4} \cdot \frac{4}{3}x^{1/3} + 4 \cdot \frac{1}{2}x^{-1/2}$

Example 18: $\int (6y^2 - 2)(8y + 5) dy$

$$\int (48y^3 + 30y^2 - 16y - 10) dy$$

$$= \frac{48y^4}{4} + 30y^3 - 16y \cdot \frac{y^2}{2} - 10y + C$$

$$= \boxed{12y^4 + 10y^3 - 8y^2 - 10y + C}$$

$$\begin{aligned}
 &\text{Note: } y = f(x) \quad \left. \begin{array}{l} f(x) = \int f'(x) dx \\ \frac{dy}{dx} = f'(x) \end{array} \right\} \\
 &\frac{dy}{dx} = f'(x) \quad \left. \begin{array}{l} y = f(x) = \int \frac{dy}{dx} dx \\ = \int dy \\ = \int 1 dy \\ = y + C \end{array} \right\} \quad 4.1.6
 \end{aligned}$$

Differential equations:

A *differential equation* is an equation involving the derivative of a function. To solve a differential equation means to find the original function.

An *initial value problem* is a common type of differential equation in which a derivative and an initial condition are given.

Example 19: Given $f'(x) = x^2 - 7$, find f . This is an example of a differential equation.

We want to find the antiderivative.

$$\begin{aligned}
 f(x) &= \int f'(x) dx \\
 &= \int (x^2 - 7) dx = \frac{x^3}{3} - 7x + C \\
 &\boxed{f(x) = \frac{x^3}{3} - 7x + C}
 \end{aligned}$$

Example 20: Suppose that $f'(x) = 3x^2 + 2\cos x$ and $f(0) = 3$. Find $f(x)$.

$$f(x) = \int f'(x) dx = \int (3x^2 + 2\cos x) dx$$

$$f(x) = \frac{3x^3}{3} + 2\sin x + C \quad \text{general antiderivative.}$$

use initial condition $f(0) = 3$ to find the specific antiderivative.

$$\begin{aligned}
 f(0) &= 3\left(\frac{0^3}{3}\right) + 2\sin 0 + C = 3 \\
 0 + 0 + C &= 3 \\
 C &= 3
 \end{aligned}$$

$$\boxed{\text{So } f(x) = x^3 + 2\sin x + 3}.$$

Example 21: Suppose that $f''(x) = 2x^3 - 6x^2 + 6x$, $f'(2) = -1$, and $f(-1) = 4$. Find $f(x)$.

$$\begin{aligned} f'(x) &= \int f''(x) dx = \int (2x^3 - 6x^2 + 6x) dx \\ &= \frac{2x^4}{4} - \frac{6x^3}{3} + \frac{6x^2}{2} + C_1 = \frac{x^4}{2} - 2x^3 + 3x^2 + C_1 \\ f'(2) &= \frac{2^4}{2} - 2(2)^3 + 3(2)^2 + C_1 = -1 \\ 8 - 16 + 12 + C_1 &= -1 \\ 4 + C_1 &= -1 \\ C_1 &= -5 \end{aligned}$$

$$f'(x) = \frac{x^4}{2} - 2x^3 + 3x^2 - 5$$

$$\begin{aligned} f(x) &= \int f'(x) dx = \int \left(\frac{x^4}{2} - 2x^3 + 3x^2 - 5 \right) dx = \frac{1}{2} \cdot \frac{x^5}{5} - 2 \cdot \frac{x^4}{4} + 3 \cdot \frac{x^3}{3} - 5x + C_2 \\ &= \frac{x^5}{10} - \frac{x^4}{2} + x^3 - 5x + C_2 \end{aligned}$$

cont'd on next page

Example 22: Suppose that $f''(x) = 12x^2 - 18x$, $f(1) = 2$, and $f(-3) = 1$. Find $f(x)$.

$$\begin{aligned} f'(x) &= \int f''(x) dx = \int (12x^2 - 18x) dx = \frac{12x^3}{3} - \frac{18x^2}{2} + C_1 \\ &= 4x^3 - 9x^2 + C_1 \end{aligned}$$

$$\begin{aligned} f(x) &= \int f'(x) dx = \int (4x^3 - 9x^2 + C_1) dx \\ &= \frac{4x^4}{4} - \frac{9x^3}{3} + C_1 x + C_2 = x^4 - 3x^3 + C_1 x + C_2 \end{aligned}$$

$$\begin{aligned} f(1) = 2 \Rightarrow f(1) &= 1^4 - 3(1)^3 + C_1(1) + C_2 = 2 \\ 1 - 3 + C_1 + C_2 &= 2 \\ -2 + C_1 + C_2 &= 2 \\ C_1 + C_2 &= 4 \end{aligned}$$

$$\begin{aligned} f(-3) = 1 \Rightarrow f(-3) &= (-3)^4 - 3(-3)^3 + C_1(-3) + C_2 = 1 \\ 81 + 81 - 3C_1 + C_2 &= 1 \\ 162 - 3C_1 + C_2 &= 1 \\ -3C_1 + C_2 &= -161 \end{aligned}$$

So we have a system of 2 equations in 2 variables : $\begin{cases} C_1 + C_2 = 4 \\ -3C_1 + C_2 = -161 \end{cases}$

Ex 21 cont'd)

$$f(x) = \frac{x^5}{5!} - \frac{x^4}{2!} + x^3 - 5x + c_2$$

$$f(-1) = 4 \Rightarrow f(-1) = \frac{(-1)^5}{5!} - \frac{(-1)^4}{2!} + (-1)^3 - 5(-1) + c_2 = 4$$

$$-\frac{1}{120} - \frac{1}{2} - 1 + 5 + c_2 = 4$$

$$-\frac{1}{120} - \frac{5}{10} + \frac{40}{10} + c_2 = 4$$

$$\frac{34}{10} + c_2 = 4$$

$$c_2 = \frac{40}{10} - \frac{34}{10} = \frac{6}{10}$$

$$c_2 = \frac{3}{5}$$

Can check against original problem.

$$f(x) = \frac{x^5}{5!} - \frac{x^4}{2!} + x^3 - 5x + \frac{3}{5}$$

Ex 22 cont'd: Solve system by elimination or substitution.

$$\begin{cases} c_1 + c_2 = 4 \\ -3c_1 + c_2 = -161 \end{cases} \xrightarrow{\text{multiply by } -1} \begin{cases} c_1 + c_2 = 4 \\ 3c_1 - c_2 = 161 \end{cases}$$

$$\text{Add: } -4c_1 = -165$$

$$c_1 = \frac{-165}{-4} = \frac{165}{4}$$

$$c_1 + c_2 = 4$$

$$c_1 = \frac{165}{4} \Rightarrow \frac{165}{4} + c_2 = 4$$

$$c_2 = 4 - \frac{165}{4} = \frac{16}{4} - \frac{165}{4} = -\frac{149}{4}$$

$$f(x) = x^4 - 3x^3 + c_1 x + c_2$$

$$f(x) = x^4 - 3x^3 + \frac{165}{4}x - \frac{149}{4}$$

$$\text{check: } f'(x) = 4x^3 - 9x^2 + \frac{165}{4}$$

$$f''(x) = 12x^2 - 18x \quad \text{OK}$$

$$f(1) = 1 - 3 + \frac{165}{4} - \frac{149}{4}$$

$$= 1 - 3 + \frac{165}{4} - \frac{149}{4} = -2 + \frac{16}{4} = -2 + 4 = 2 \quad \text{OK}$$

Check
cont'd
next page

Ex 22 cont'd (checkng):

$$f(-3) = (-3)^4 - 3(-3)^3 + \frac{165}{4}(-3) - \frac{149}{4} = 81 - 3(-27) - \frac{495}{4} - \frac{149}{4}$$

$$= 81 + 81 - \frac{644}{4} = 162 - 161 = 1 \quad \text{OK}$$

4.1.8
+ check

Velocity and acceleration (rectilinear motion):

We already know that if $f(t)$ is the position of an object at time t , then $f'(t)$ is its velocity and $f''(t)$ is its acceleration.

Note: Acceleration due to gravity near the earth's surface is approximately 9.8 m/s^2 or 32 ft/s^2 .

Example 23: Suppose a particle's velocity is given by $v(t) = 2\sin t + \cos t$ and its initial position is $s(0) = 3$. Find the position function of the particle.

$$v(t) = s'(t) = 2\sin t + \cos t$$

$$\begin{aligned} s(t) &= \int v(t) dt = \int s'(t) dt = \int (2\sin t + \cos t) dt \\ &= 2(-\cos t) + \sin t + C = -2\cos t + \sin t + C \end{aligned}$$

$$s(0) = 3 \Rightarrow s(0) = -2\cos(0) + \sin(0) + C = 3$$

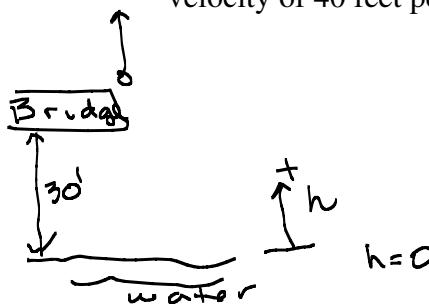
$$-2(1) + 0 + C = 3$$

$$-2 + C = 3$$

$$C = 5$$

$$\boxed{s(t) = -2\cos t + \sin t + 5}$$

Example 24: Suppose a ball is thrown upward from a 30-foot bridge over a river at an initial velocity of 40 feet per second. How high does it go? When does it hit the water?



Let h = distance above the water

acceleration: $a(t) = h''(t) = -32$

$$\begin{aligned} \text{velocity: } v(t) &= h'(t) = \int h''(t) dt \\ &= \int -32 dt \\ &= -32t + C_1 \end{aligned}$$

$$\text{Initial velocity: } v(0) = +40 \Rightarrow -32(0) + C_1 = 40$$

$$C_1 = 40$$

$$\text{Velocity: } v(t) = h'(t) = -32t + 40$$

$$\begin{aligned} \text{Position: } h(t) &= \int v(t) dt = \int h'(t) dt = \int (-32t + 40) dt \\ &= -\frac{32t^2}{2} + 40t + C_2 = -16t^2 + 40t + C_2 \end{aligned}$$

$$\text{Initial height: } h(0) = 30 \Rightarrow h(0) = -16(0)^2 + 40(0) + C_2 = 30$$

$$C_2 = 30$$

$$\boxed{\text{Position: } h(t) = -16t^2 + 40t + 30}$$

Note: If the initial velocity is v_0 and initial position is h_0 , then the position function is

$$h(t) = -\frac{at^2}{2} + v_0 t + h_0$$

where a is the acceleration due to gravity.

Ex 24 cont'd:

$$h(t) = -16t^2 + 40t + 30$$

How high does it go? Set $v(t) = 0$.

$$v(t) = 0 \Rightarrow -32t + 40 = 0$$

$$40 = 32t$$

$$\frac{40}{32} = t$$

$$t = \frac{5}{4} = 1.25 \text{ sec}$$

$$h\left(\frac{5}{4}\right) = -16\left(\frac{5}{4}\right)^2 + 40\left(\frac{5}{4}\right) + 30$$

$$= -16\left(\frac{25}{16}\right) + 50 + 30 = -25 + 80 = \boxed{55 \text{ ft}}$$

max height

When does it hit the water? Set $h(t) = 0$:

$$-16t^2 + 40t + 30 = 0$$

$$-2(8t^2 - 20t - 15) = 0$$

Quadratic formula:

$$t = \frac{20 \pm \sqrt{(-20)^2 - 4(8)(-15)}}{2(8)}$$

$$= \frac{20 \pm \sqrt{880}}{16}$$

$$= \frac{20 + \sqrt{880}}{16}, \frac{20 - \sqrt{880}}{16}$$

check

$$\approx 3.104 \text{ sec}, -0.604 \text{ sec}$$

$$h(3.104) = -16(3.104)^2 + 40(3.104) + 30$$

= 0
it hits the water at $t = 3.104 \text{ sec}$