

## 4.4: The Fundamental Theorem of Calculus

Evaluating the area under a curve by calculating the areas of rectangles, adding them up, and letting taking the limit as  $n \rightarrow \infty$  is okay in theory but is tedious at best and not very practical.

Fortunately, there is a theorem that makes calculating the area under the curve (definite integral) much easier.

The Fundamental Theorem of Calculus:

Let  $f$  be continuous on the interval  $[a, b]$ . Then,

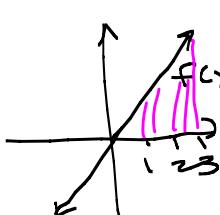
$$\int_a^b f(x) dx = F(b) - F(a)$$

where  $F$  is any antiderivative of  $f$ ; in other words, where  $F'(x) = f(x)$ .

Notation: We'll use this notation when evaluating definite integrals.

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

Example 1: Find the area under the graph of  $f(x) = x$  between 0 and 3.



$$\text{Area} = \int_0^3 x \, dx = \frac{x^2}{2} \Big|_0^3 = \frac{3^2}{2} - \frac{0^2}{2} = \frac{9}{2} - 0 = \boxed{\frac{9}{2}}$$

Note:  $\int x \, dx = \frac{x^2}{2} + C$

$$\begin{aligned} \int_0^3 x \, dx &= \left( \frac{x^2}{2} + C \right) \Big|_0^3 = \left( \frac{3^2}{2} + C \right) - \left( \frac{0^2}{2} + C \right) \\ &= \frac{9}{2} + C - 0 - C = \boxed{\frac{9}{2}} \end{aligned}$$

Notice that the constant  $C$  disappeared when we evaluated the definite integral. This will always happen.

$$\int_a^b f(x)dx = F(x) + C \Big|_a^b = (F(b) + C) - (F(a) + C) = F(b) - F(a)$$

So from now on, we'll omit the "+c" when evaluating definite integrals.

**Example 2:** Find the area under the graph of  $f(x) = 4x^2 + 1$  over the interval  $[0, 2]$ . (Compare with our approximation in Section 4.2, Example 5).

$$\begin{aligned} \text{Area} &= \int_0^2 (4x^2 + 1) dx = \left( \frac{4x^3}{3} + x \right) \Big|_0^2 \\ &= \left( \frac{4(2)^3}{3} + 2 \right) - \left( \frac{4(0)^3}{3} + 0 \right) \\ &= \frac{32}{3} + 2 - 0 = \frac{32}{3} + \frac{6}{3} = \boxed{\frac{38}{3}} = \boxed{12\frac{2}{3}} \end{aligned}$$

**Example 3:** Evaluate  $\int_{-2}^4 (3x^2 - x + 4)dx$ .

Left approximation was 9  
Right approx was 17  
Midpoint approximation 12.5

$$\begin{aligned} &\int_{-2}^4 (3x^2 - x + 4) dx \\ &= \left( \frac{3x^3}{3} - \frac{x^2}{2} + 4x \right) \Big|_{-2}^4 = \left( x^3 - \frac{x^2}{2} + 4x \right) \Big|_{-2}^4 \\ &= \left( 4^3 - \frac{4^2}{2} + 4(4) \right) - \left( (-2)^3 - \frac{(-2)^2}{2} + 4(-2) \right) \\ &= (64 - 8 + 16) - (-8 - 2 - 8) \\ &= 72 - (-18) \\ &= 72 + 18 = \boxed{90} \end{aligned}$$

Note: Check:  $\frac{d}{dx} \left( x^3 - \frac{x^2}{2} + 4x \right)$

$$\begin{aligned} &= 3x^2 - \frac{1}{2}(2x) + 4 \\ &= 3x^2 - x + 4 \quad \checkmark \text{OK} \end{aligned}$$

Example 4: Evaluate  $\int_0^\pi (4x^3 + \cos x) dx$ .

$$\begin{aligned} \int_0^\pi (4x^3 + \cos x) dx &= \left( \frac{4x^4}{4} + \sin x \right) \Big|_0^\pi \\ &= (x^4 + \sin x) \Big|_0^\pi = (\pi^4 + \sin \pi) - (0^4 + \sin 0) \\ &= \pi^4 + 0 - 0 + 0 = \boxed{\pi^4} \end{aligned}$$

Example 5: Evaluate  $\int_1^3 \left( \frac{3}{t^2} \right) dt$ .

$$\begin{aligned} \int_1^3 \left( \frac{3}{t^2} \right) dt &= 3 \int_1^3 t^{-2} dt = 3 \frac{t^{-1}}{-1} \Big|_1^3 = -\frac{3}{t} \Big|_1^3 \\ &= -\frac{3}{3} - \left( -\frac{3}{1} \right) = -1 + 3 = \boxed{2} \end{aligned}$$

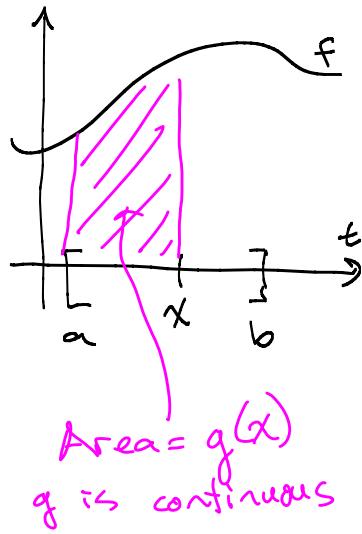
Example 6: Evaluate  $\int_2^9 \frac{1}{\sqrt{u}} du$ .

$$\begin{aligned} \int_2^9 \frac{1}{\sqrt{u}} du &= \int_2^9 u^{-\frac{1}{2}} du = \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \Big|_2^9 \\ &= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \Big|_2^9 = 2u^{\frac{1}{2}} \Big|_2^9 = 2\sqrt{u} \Big|_2^9 \\ &= 2\sqrt{9} - 2\sqrt{2} = 2(3) - 2\sqrt{2} = \boxed{6 - 2\sqrt{2}} \end{aligned}$$

Example 7: Evaluate  $\int_{-2}^4 \frac{1}{x^3} dx$

improper integral

For now, if  $f$  has an infinite discontinuity anywhere in  $[a, b]$ , assume that  $\int_a^b f(x) dx$  does not exist. Some of these integrals do exist....you will learn how to handle such integrals in Calculus 2.



### The Fundamental Theorem of Calculus, Part II:

Let  $f$  be continuous on the interval  $[a, b]$ . Then the function  $g$  defined by

$$g(x) = \int_a^x f(t) dt, \quad a \leq x \leq b$$

is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and  $g'(x) = f(x)$ .

$$\text{In other words, } \frac{d}{dx} \left[ \int_a^x f(t) dt \right] = f(x).$$

$g'(x) =$   
rate of  
change in  
Area

Example 1: Find the derivative of the function  $g(x) = \int_3^x \frac{t^2 - 2t + 4}{t-2} dt$ .

$$g'(x) = \frac{d}{dx} \int_3^x \frac{t^2 - 2t + 4}{t-2} dt$$

$$= f(x) = \boxed{\frac{x^2 - 2x + 4}{x-2}}$$

Note:  $f(t) = \frac{t^2 - 2t + 4}{t-2}$  equal to value of  $f(x)$

is discontinuous at  $t=2$   
but continuous on

$[3, x]$  if  
 $x \geq 3$

Example 2: Find  $\frac{d}{dx} \int_{-2}^{\sin x} \sqrt{t^4 + 2} dt$ .

$$\text{Area} = A = \int_{-2}^{\sin x} \sqrt{t^4 + 2} dt.$$

I want to find  $\frac{dA}{dx}$

Chain rule:  $\frac{dA}{dx} = \frac{dA}{du} \cdot \frac{du}{dx}$

Let  $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$\text{Note: } f(t) = \sqrt{t^4 + 2}$$

is continuous on  $(-\infty, \infty)$

$$\frac{dA}{du} = \boxed{\int_{-2}^u \sqrt{t^4 + 2} dt}$$

$$\frac{dA}{du} = \sqrt{u^4 + 2}$$

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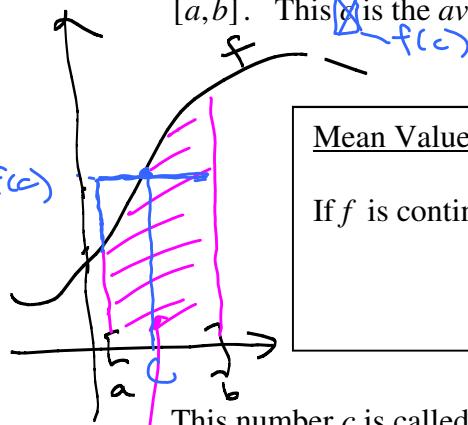
$$\frac{dA}{dx} = \frac{dA}{du} \cdot \frac{du}{dx} = \sqrt{u^2 + 2} (\cos x) \\ = \sqrt{(\sin x)^2 + 2} (\cos x)$$

4.4.5

The mean (average) value of a function:

$$\boxed{\cos x \int \sin^2 x + 2}$$

On the interval  $[a, b]$ , a continuous function  $f(x)$  will have an average "height"  $c$  such that the rectangle with width  $b - a$  and height  $c$  will have the same area as the area under the curve over  $[a, b]$ . This  ~~$\square$~~  is the *average value of the function  $f$  over  $[a, b]$* .



$f(c)$  is the average value of the function

#### Mean Value Theorem for Integrals:

If  $f$  is continuous on  $[a, b]$ , then there exists a number  $c$  in  $[a, b]$  such that

$$\int_a^b f(x) dx = f(c)(b - a).$$

This number  $c$  is called the *average value* of the function  $f$  on the interval  $[a, b]$ .

$$\text{Area} = \int_a^b f(x) dx$$

The *average value* of a continuous function  $f$  on the interval  $[a, b]$  is given by

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

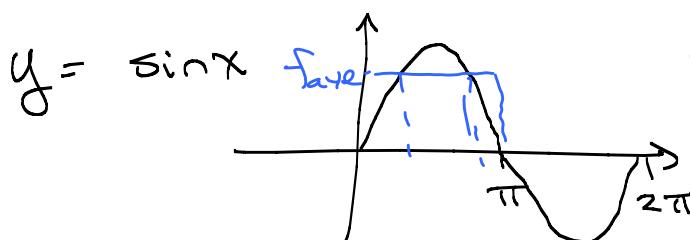
$$\hookrightarrow (b-a)f_{\text{ave}} = \int_a^b f(x) dx$$

Example 8: Find the average value of the function  $f(x) = 4x^3 - x^2$  over the interval  $[-3, 2]$ .

$$\text{width} = b - a = 2 - (-3) = 5$$

$$\begin{aligned} \text{(Net) Area under curve} &= \int_{-3}^2 (4x^3 - x^2) dx = \left( \frac{4x^4}{4} - \frac{x^3}{3} \right) \Big|_{-3}^2 \\ &= \left( x^4 - \frac{x^3}{3} \right) \Big|_{-3}^2 = \left( 2^4 - \frac{2^3}{3} \right) - \left( (-3)^4 - \frac{(-3)^3}{3} \right) \\ &= \left( 16 - \frac{8}{3} \right) - \left( 81 - \frac{-27}{3} \right) = 16 - \frac{8}{3} - 81 + 9 = -74 - \frac{8}{3} \\ &\quad - \frac{222}{3} - \frac{8}{3} = -\frac{230}{3} \end{aligned}$$

Example 9: Determine the average value of  $f(x) = \sin x$  on the interval  $[0, \pi]$ .



$$f_{\text{ave}} = \frac{1}{\text{width}} \text{ (Net Area)}$$

$$= \frac{1}{\pi - 0} \int_0^\pi \sin x dx$$

$$= \frac{1}{\pi} (-\cos x) \Big|_0^\pi = \frac{1}{\pi} (-\cos \pi - (-\cos 0))$$

$$= \frac{1}{\pi} (-\cos \pi + \cos 0) = \frac{1}{\pi} (-(-1) + 1) = \frac{1}{\pi} (2) = \frac{2}{\pi}$$

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{\pi} \left( -\frac{230}{3} \right) \\ &= -\frac{230}{\pi} = -\frac{46}{3} \end{aligned}$$

$$f_{\text{ave}} = \frac{\text{area under curve}}{\text{width}}$$