

5.1: The Natural Logarithmic Function: Differentiation

An algebraic approach to logarithms:

Definition: $\log_b x = y$ is equivalent to $b^y = x$.

The functions $f(x) = b^x$ and $g(x) = \log_b x$ are inverses of each other.
 b is called the *base* of the logarithm.

The logarithm of base e is called the *natural logarithm*, which is abbreviated “ln”.

The natural logarithm:

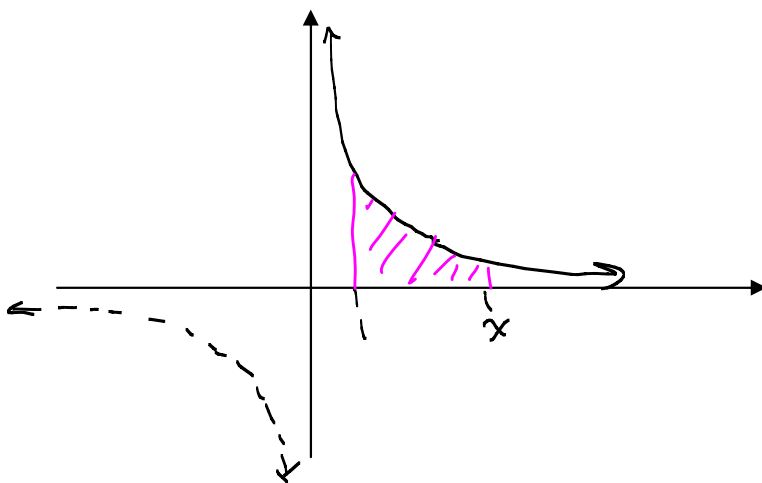
$$\ln x = \log_e x.$$

Therefore $\ln x = y$ is equivalent to $e^y = x$ and the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverses of each other.

A calculus approach to the natural logarithm:

The natural logarithm function is defined as

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0.$$



$$f(t) = \frac{1}{t}$$

For $x > 1$, $\ln x$ can be interpreted as the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to $t = x$.

Note: The integral is not defined for $x < 0$.

$$\text{For } x = 1, \ln x = \int_1^1 \frac{1}{t} dt = 0.$$

$$\text{For } x < 1, \ln x = \int_1^x \frac{1}{t} dt = -\int_x^1 \frac{1}{t} dt < 0.$$

Recall:

The Fundamental Theorem of Calculus, Part II:

Let f be continuous on the interval $[a, b]$. Then the function g defined by

$$g(x) = \int_a^x f(t) dt, \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b) , and $g'(x) = f(x)$.

Apply the Fundamental Theorem of Calculus to the function $f(t) = \frac{1}{t}$.

$$\frac{d}{dx} \left(\int_1^x \frac{1}{t} dt \right) = \frac{1}{x} \quad \frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{d}{dx} (\ln x) = \frac{1}{x}$$

This means that $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

The Derivative of the Natural Logarithmic Function

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Laws of Logarithms:

If x and y are positive numbers and r is a rational number, then:

$$1. \ln(xy) = \ln x + \ln y$$

$$2. \ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

Note: This also gives us $\ln\left(\frac{1}{x}\right) = -\ln x$.

$$3. \ln(x^r) = r \ln x$$

Example 1: Expand $\ln\left(\frac{x^3\sqrt{x+5}}{x^2+4}\right)$.

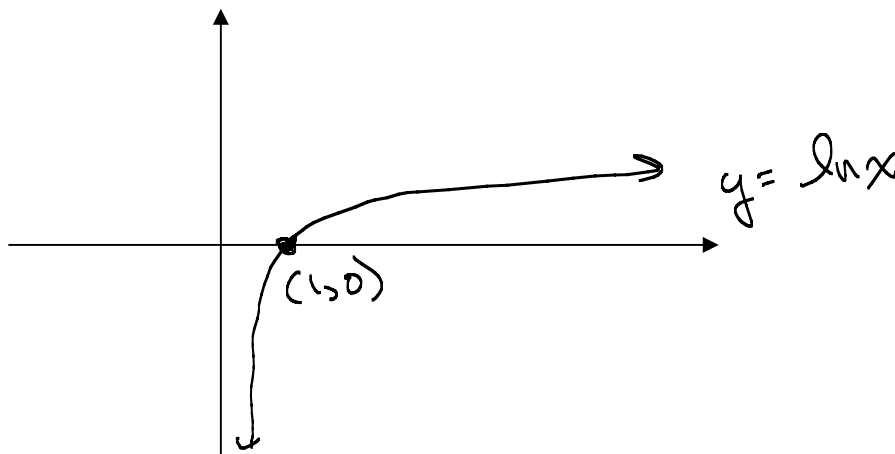
$$\begin{aligned}
 &= \ln \frac{x^3(x+5)^{1/2}}{x^2+4} \\
 &= \ln [x^3(x+5)^{1/2}] - \ln(x^2+4) \\
 &= \ln x^3 + \ln(x+5)^{1/2} - \ln(x^2+4) \\
 &= 3\ln x + \frac{1}{2}\ln(x+5) - \ln(x^2+4)
 \end{aligned}$$

The graph of $y = \ln x$:

It can be shown that $\lim_{x \rightarrow \infty} \ln x = \infty$ and that $\lim_{x \rightarrow 0^+} \ln x = -\infty$.

For $x > 0$, $\frac{dy}{dx} = \frac{1}{x} > 0$ so $y = \ln x$ is increasing on $(0, \infty)$.

For $x > 0$, $\frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$ so $y = \ln x$ is concave down on $(0, \infty)$.



Because $\ln 1 = 0$ and $y = \ln x$ is increasing to arbitrarily large values ($\lim_{x \rightarrow \infty} \ln x = \infty$), the

Intermediate Value Theorem guarantees that there is a number x such that $\ln x = 1$. That number is called e .

$$e \approx 2.71828182845904523536$$

(e is an irrational number—it cannot be written as a decimal that ends or repeats.)

$$\frac{d}{dx} (\ln x) = \frac{1}{x}$$

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Example 2: Find $\frac{dy}{dx}$ for $y = \ln(2x^5 + 3x)$.

$$\frac{dy}{dx} = \frac{1}{2x^5 + 3x} \frac{d}{dx} (2x^5 + 3x)$$

$$= \frac{1}{2x^5 + 3x} (10x^4 + 3) =$$

$$\boxed{\frac{10x^4 + 3}{2x^5 + 3x}}$$

Ex: $y = \cos(2x^5 + 3x)$

$$\frac{dy}{dx} = -\sin(2x^5 + 3x) \cdot \frac{d}{dx} (2x^5 + 3x)$$

$$= -(\sin(2x^5 + 3x))(10x^4 + 3)$$

Note: $\frac{d}{dx}(\ln u) = \frac{1}{u} \frac{du}{dx}$ or, written another way, $\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$.

Example 3: Determine $\frac{d}{dx}(\ln(\cos x))$.

$$\frac{d}{dx} (\ln(\cos x)) = \frac{1}{\cos x} \frac{d}{dx} (\cos x)$$

$$= \frac{1}{\cos x} (-\sin x) = -\frac{\sin x}{\cos x} = \boxed{-\tan x}$$

Example 4: Find the derivative of $f(x) = \frac{1}{\ln x}$.

$$f(x) = \frac{1}{\ln x} = (\ln x)^{-1} = -1(\ln x)^{-2} \frac{d}{dx} (\ln x)$$

$$= -1(\ln x)^{-2} \left(\frac{1}{x}\right) = \boxed{-\frac{1}{x(\ln x)^2}}$$

Example 5: Find the derivative of $f(x) = x^2 \ln x$.

Product Rule: $f'(x) = x^2 \frac{d}{dx} (\ln x) + (\ln x) \frac{d}{dx} (x^2)$

$$= x^2 \left(\frac{1}{x}\right) + (\ln x)(2x)$$

$$= \boxed{x + 2x \ln x}$$

Example 6: Find the derivative of $y = \frac{\ln x}{4x}$.

Quotient Rule: $\frac{dy}{dx} = \frac{(4x) \frac{d}{dx} (\ln x) - (\ln x) \frac{d}{dx} (4x)}{(4x)^2}$

$$= \frac{4x \left(\frac{1}{x}\right) - (\ln x)(4)}{16x^2} =$$

$$\frac{4 - 4 \ln x}{16x^2}$$

$$= \frac{4(1 - \ln x)}{16x^2}$$

$$= \boxed{\frac{1 - \ln x}{4x^2}}$$

Example 7: Find the derivative of $g(t) = \ln(7t)$.

$$g'(t) = \frac{1}{7t} \frac{d}{dt}(7t) = \frac{1}{7t} \cdot 7 = \frac{7}{7t} = \boxed{\frac{1}{t}}$$

Note: $g(t) = \ln(7t) = \ln(t) + \ln(7)$

$$g'(t) = \frac{1}{t} + 0 = \boxed{\frac{1}{t}}$$

Example 8: Determine the derivative of $f(x) = \frac{\ln 6x}{(x+4)^5}$.

$$\begin{aligned} f'(x) &= \frac{(x+4)^5 \frac{d}{dx}(\ln(6x)) - (\ln(6x)) \frac{d}{dx}(x+4)^5}{(x+4)^{10}} \\ &= \frac{(x+4)^5 \left(\frac{1}{6x}\right)(6) - (\ln(6x))(5)(x+4)^4(1)}{(x+4)^{10}} \\ &= \frac{(x+4)^5 \left(\frac{1}{x}\right) - 5(x+4)^4 \ln(6x)}{(x+4)^6} \\ &= \frac{(x+4)^4 \left[(x+4) \left(\frac{1}{x}\right) - 5 \ln(6x) \right]}{(x+4)^6} = \frac{(x+4) \left(\frac{1}{x}\right) - 5 \ln(6x)}{(x+4)^6} \left(\frac{x}{x}\right) \\ &= \boxed{\frac{x+4 - 5x \ln(6x)}{x(x+4)^6}} \end{aligned}$$

Logarithmic differentiation:

To differentiate $y = f(x)$:

1. Take the natural logarithm of both sides.
2. Use the laws of logarithms to expand.
3. Differentiate implicitly with respect to x .
4. Solve for $\frac{dy}{dx}$.

Example 9: Use logarithmic differentiation to find the derivative of

$$y = (x^2 + 2)^5 (2x + 1)^3 (6x - 1)^2.$$

$$\ln y = \ln \left[(x^2 + 2)^5 (2x + 1)^3 (6x - 1)^2 \right]$$

$$\ln y = 5 \ln(x^2 + 2) + 3 \ln(2x + 1) + 2 \ln(6x - 1)$$

$$\frac{d}{dx} (\ln y) = \frac{d}{dx} \left[5 \ln(x^2 + 2) + 3 \ln(2x + 1) + 2 \ln(6x - 1) \right]$$

$$\frac{1}{y} \frac{dy}{dx} = 5 \left(\frac{1}{x^2 + 2} \right) (2x) + 3 \left(\frac{1}{2x + 1} \right) (2) + 2 \left(\frac{1}{6x - 1} \right) (6)$$

$$\frac{dy}{dx} = y \left[\frac{10x}{x^2 + 2} + \frac{6}{2x + 1} + \frac{12}{6x - 1} \right]$$

Example 10: Find y' for $y = \frac{(x^3 + 1)^4 \sin^2 x}{\sqrt[3]{x}}$.

$$\frac{dy}{dx} = \left[(x^3 + 1)^5 (2x + 1)^3 (6x - 1)^2 \left[\frac{10x}{x^2 + 2} + \frac{6}{2x + 1} + \frac{12}{6x - 1} \right] \right]$$

$$y = \frac{(x^3 + 1)^4 (\sin x)^2}{x^{1/3}}$$

$$\ln y = \ln \frac{(x^3 + 1)^4 (\sin x)^2}{x^{1/3}}$$

$$\ln y = 4 \ln(x^3 + 1) + 2 \ln(\sin x) - \frac{1}{3} \ln x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 4 \left(\frac{1}{x^3 + 1} \right) (3x^2) + 2 \left(\frac{1}{\sin x} \right) (\cos x) - \frac{1}{3} \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y \left[\frac{12x^2}{x^3 + 1} + 2 \cot x - \frac{1}{3x} \right]$$

$$= \left[\frac{(x^3 + 1)^4 \sin^2 x}{\sqrt[3]{x}} \left[\frac{12x^2}{x^3 + 1} + 2 \cot x - \frac{1}{3x} \right] \right]$$