

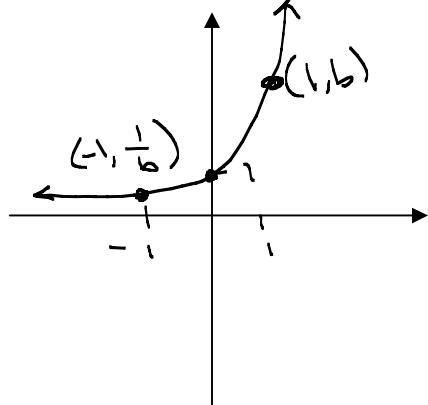
## 5.4: Exponential Functions: Differentiation and Integration

### Short Review:

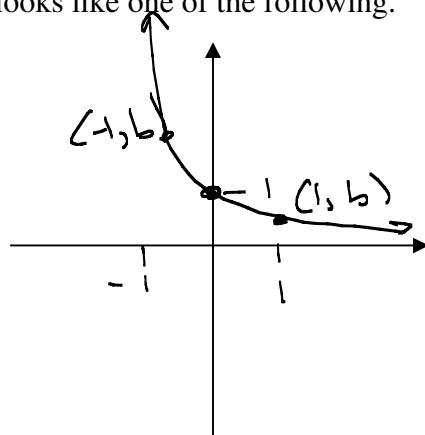
An *exponential* function takes the form  $f(x) = b^x$ , where  $b > 0$  and  $b \neq 1$ .

For any exponential function  $f(x) = b^x$ , the graph looks like one of the following.

$$\begin{array}{|c|l|} \hline x & y = b^x \\ \hline 0 & b^0 = 1 \\ 1 & b^1 = b \\ -1 & b^{-1} = \frac{1}{b} \\ \hline \end{array}$$



$$b > 1$$



$$0 < b < 1$$

Notice:

- Domain is  $(-\infty, \infty)$ .
- Range is  $(0, \infty)$ .
- Horizontal asymptote is  $y = 0$ .
- Always passes through the points  $(0, 1)$ ,  $(1, b)$ ,  $(-1, \frac{1}{b})$

### The natural exponential function:

The number  $e$  can be defined in several ways.

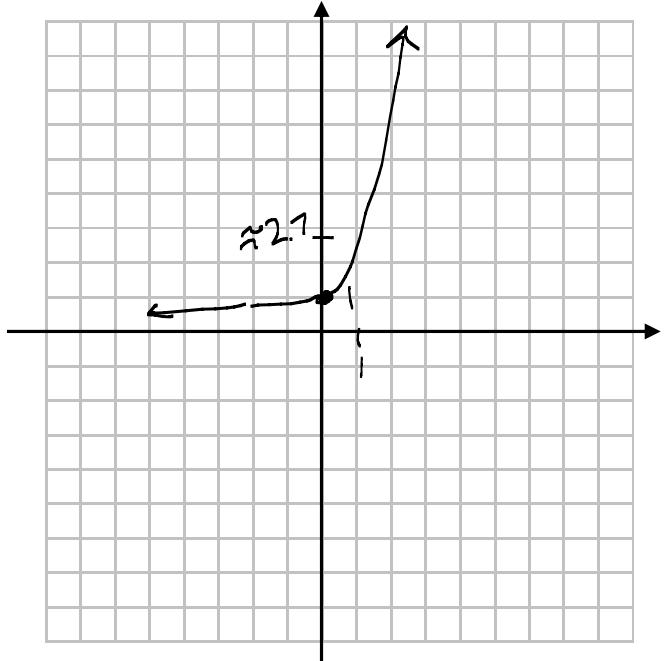
One definition of the number  $e$ :

$e$  is the number such that  $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$

$$e \approx 2.718281828459$$

The slope of the tangent line at the point  $(0,1)$  is equal to 1.

**The graph of  $f(x) = e^x$ :**



Another definition of the number  $e$ :

$$e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \quad \text{or, equivalently, } e = \lim_{x \rightarrow 0} (1+x)^{1/x}$$

**Derivatives of exponential functions:**

$$\frac{d}{dx}(e^x) = e^x$$

**Example 1:** Find the derivative of  $f(x) = -7e^x$ .

$$f'(x) = \overbrace{-7e^x}^{(-7e^x)}$$

Example 2: Find the derivative of  $f(x) = 5\sqrt{e^x + 7}$ .

$$f(x) = 5\sqrt{e^x + 7} = 5(e^x + 7)^{1/2}$$

$$f'(x) = 5 \left(\frac{1}{2}\right)(e^x + 7)^{-1/2} \frac{d}{dx}(e^x + 7) = \frac{5}{2}(e^x + 7)^{-1/2}(e^x)$$

Example 3: Find the derivative of  $f(x) = e^x \sin x$ .

$$f'(x) = e^x \frac{d}{dx}(\sin x) + (\sin x) \frac{d}{dx}(e^x)$$

$$= e^x \cos x + (\sin x)e^x = e^x \cos x + e^x \sin x$$

$$= \frac{5e^x}{2\sqrt{e^x + 7}}$$

Example 4: Find the derivative of  $g(x) = e^{-7x} + 2x^3 - 4e$ .

$$g'(x) = e^{-7x} \frac{d}{dx}(-7x) + 6x^2 + 0$$

$$= e^{-7x}(-7) + 6x^2 = -7e^{-7x} + 6x^2$$

Example 5: Find the derivative of  $y = e^{x^2+4x}$ .

$$\frac{dy}{dx} = e^{x^2+4x} \frac{d}{dx}(x^2 + 4x) = e^{x^2+4x}(2x + 4)$$

Example 6: Find the derivative of  $f(x) = \cos(e^x - x)$ .

$$f'(x) = -\sin(e^x - x) \frac{d}{dx}(e^x - x)$$

$$= [-\sin(e^x - x)][e^x - 1] = -(e^x - 1)\sin(e^x - x)$$

Example 7: Find the equation of the tangent line to the graph of  $f(x) = (e^x + 2)^2$  at the point  $(0, 9)$ .

$$f'(x) = 2(e^x + 2) \frac{d}{dx}(e^x + 2)$$

$$= 2(e^x + 2)(e^x + 0) = 2e^x(e^x + 2)$$

$$\text{slope: } m = f'(0) = 2e^0(e^0 + 2) = 2(1)(1 + 2) = 2(3) = 6$$

$$\text{check/find point: } f(0) = (e^0 + 2)^2 = (1 + 2)^2 = 3^2 = 9 \quad \checkmark$$

$$y - y_1 = m(x - x_1)$$

or

$$y - 9 = 6(x - 0)$$

$$y - 9 = 6x$$

$$y = 6x + 9$$

$$y = mx + b$$

$$y = 6x + 9$$

## Integration of exponential functions:

$$\int e^x dx = e^x + C$$

Example 8: Determine  $\int (x^2 - 5e^x) dx$

$$\int (x^2 - 5e^x) dx = \boxed{\frac{x^3}{3} - 5e^x + C}$$

Example 9: Find  $\int e^{5t} dt$ .

$$\begin{aligned} \int e^{5t} dt &= \frac{1}{5} \int e^u du \\ &= \frac{1}{5} e^u + C = \boxed{\frac{1}{5} e^{5t} + C} \end{aligned}$$

$$u = 5t$$

$$\frac{du}{dt} = 5$$

$$du = 5 dt$$

$$\frac{1}{5} du = dt$$

Example 10: Find  $\int_1^3 e^{2x-3} dx$ .

$$\begin{aligned} \int_1^3 e^{2x-3} dx &= \frac{1}{2} \int_{u=-1}^{u=3} e^u du \\ &= \frac{1}{2} e^u \Big|_{u=-1} = \boxed{\frac{1}{2} [e^3 - e^{-1}]} \end{aligned}$$

or (without changing limits of integration)

$$\begin{aligned} \int_1^3 e^{2x-3} dx &= \frac{1}{2} \int_{x=1}^{x=3} e^u du = \frac{1}{2} e^u \Big|_{x=1}^{x=3} \\ &= \frac{1}{2} e^{2x-3} \Big|_{x=1}^{x=3} = \frac{1}{2} \left[ e^{2(3)-3} - e^{2(1)-3} \right] \end{aligned}$$

$$\begin{aligned} u &= 2x-3 \\ \frac{du}{dx} &= 2 \\ du &= 2 dx \\ \frac{1}{2} du &= dx \\ x=1 &\Rightarrow u=2(1)-3=-1 \\ x=3 &\Rightarrow u=2(3)-3=3 \\ &= 6-3=3 \end{aligned}$$

$$\boxed{\frac{1}{2} [e^3 - e^{-1}]}$$

Example 11: Find  $\int te^{t^2} dt$ .

$$\begin{aligned} \int t e^{t^2} dt &= \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + C \\ &= \boxed{\frac{1}{2} e^{t^2} + C} \end{aligned}$$

$$u = t^2$$

$$\frac{du}{dt} = 2t$$

$$du = 2t dt$$

$$\frac{1}{2} du = t dt$$

Example 12: Determine  $\int \frac{e^x}{\sqrt[3]{e^x + 1}} dx$ .

$$\begin{aligned} \int e^x (e^x + 1)^{-\frac{1}{3}} dx &= \int u^{-\frac{1}{3}} du \\ &= \frac{u^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C = \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + C = \frac{3u^{\frac{2}{3}}}{2} + C \\ &= \boxed{\frac{3}{2} (e^x + 1)^{\frac{2}{3}} + C} \end{aligned}$$

$$u = e^x + 1$$

$$\frac{du}{dx} = e^x$$

$$du = e^x dx$$

Example 13: Determine  $\int \frac{e^x - e^{-x}}{e^{3x}} dx$

$$\begin{aligned} \int \left( \frac{e^x}{e^{3x}} - \frac{e^{-x}}{e^{3x}} \right) dx &= \int (e^{x-3x} - e^{-x-3x}) dx \\ &= \int (e^{-2x} - e^{-4x}) dx = \int e^{-2x} dx - \int e^{-4x} dx \\ &= -\frac{1}{2} \int e^u du - \left( -\frac{1}{4} \right) \int e^{u_2} du_2 \\ &= -\frac{1}{2} e^u + \frac{1}{4} e^{u_2} + C \\ &= \boxed{\frac{1}{2} e^{-2x} + \frac{1}{4} e^{-4x} + C} \end{aligned}$$

$$\begin{aligned} \text{1st one: } u &= -2x \\ du &= -2 dx \\ -\frac{1}{2} du &= dx \end{aligned}$$

2nd one:

$$\begin{aligned} u_2 &= -4x \\ du_2 &= -4 dx \\ -\frac{1}{4} du_2 &= dx \end{aligned}$$