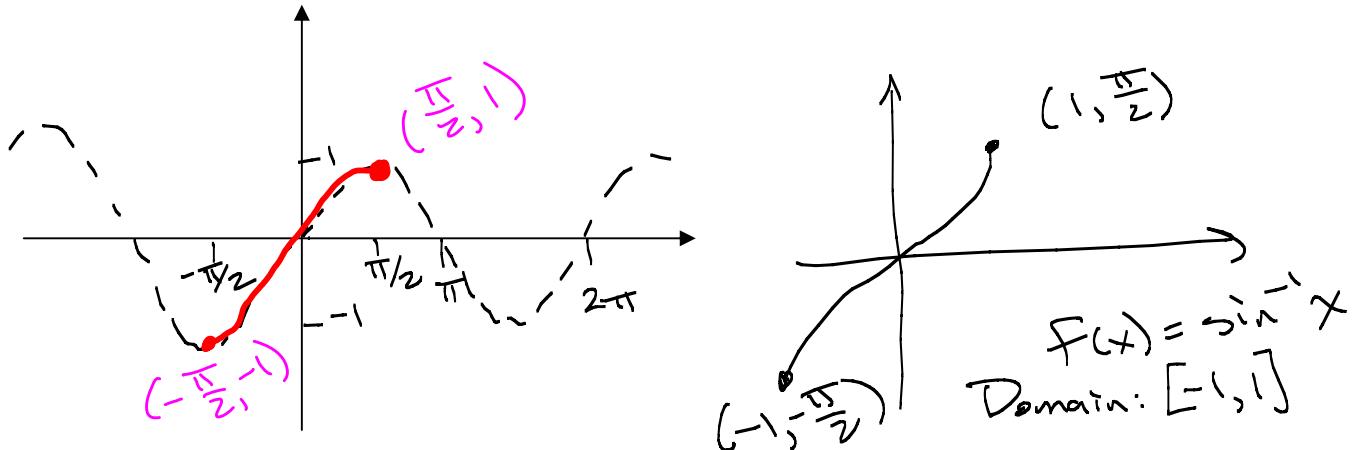


5.6: Inverse Trigonometric Functions – Differentiation

Because none of the trigonometric functions are one-to-one, none of them have an inverse function. To overcome this problem, the domain of each function is restricted so as to produce a one-to-one function.

Inverse sine function:

$$y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x \quad \text{and} \quad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

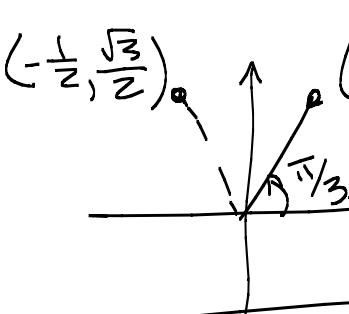


Properties of the inverse sine function:

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\sin(\sin^{-1} x) = x \quad \text{for } -1 \leq x \leq 1$$

Example 1: Evaluate $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\sin^{-1}\left(-\frac{1}{2}\right)$.



$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

$$\text{So, } \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\sin^{-1}\left(-\frac{1}{2}\right)$$

$$\sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$$

but $\frac{11\pi}{6} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Ex 1: Find $\sin^{-1}(-1.5)$. $\sin \theta = -1 \Rightarrow$ is impossible
 $\sin^{-1}(-\frac{3}{2})$ $-1 \leq \sin \theta \leq 1$
5.6.2
 $\therefore \sin^{-1}(-1.5)$ is undefined.

Example 2: Evaluate $\cot(\sin^{-1}\frac{3}{7})$

Example 3: Evaluate $\sin(\sin^{-1}(-0.54))$.

-0.54

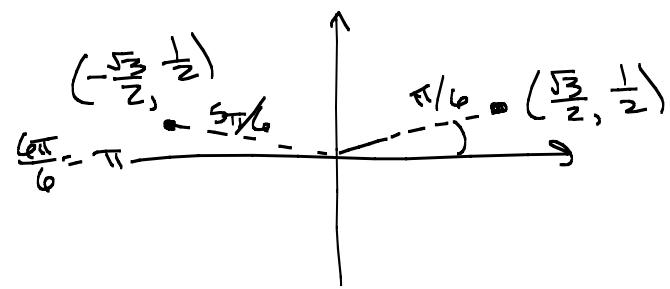
Example 4: Evaluate $\sin(\sin^{-1} 2)$.

undefined

$2 \notin [-1, 1]$

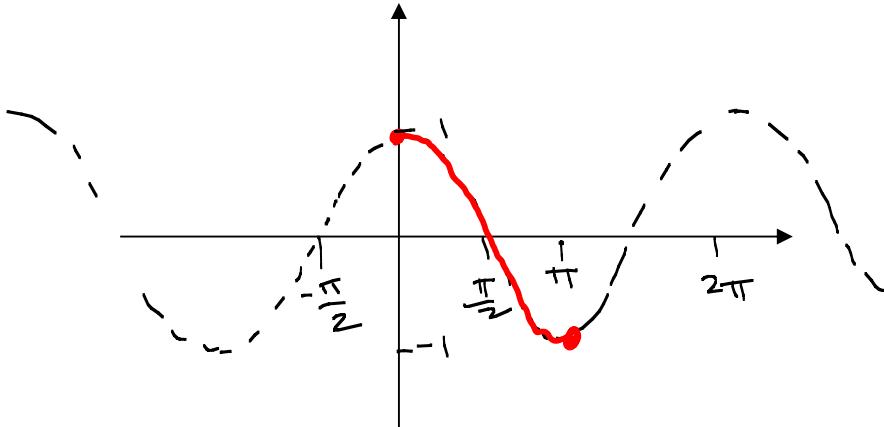
Example 5: Evaluate $\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

Example 6: Evaluate $\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$



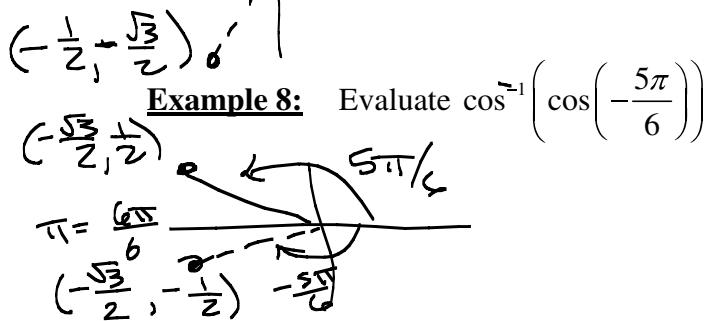
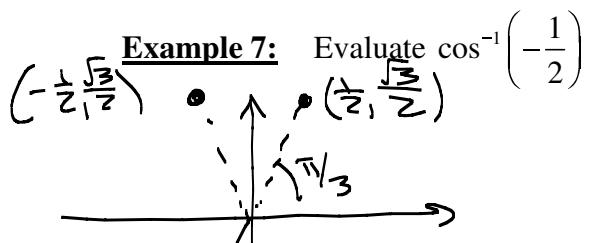
Inverse cosine function:

$$y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x \quad \text{and} \quad y \in [0, \pi]$$

Properties of the inverse cosine function:

$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leq x \leq \pi$$

$$\cos(\cos^{-1} x) = x \quad \text{for } -1 \leq x \leq 1$$



$$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\cos \theta = -\frac{1}{2} \quad \text{and} \quad \theta \in [0, \pi]$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \boxed{\frac{2\pi}{3}}$$

$$\cos^{-1}(\cos(-\frac{5\pi}{6}))$$

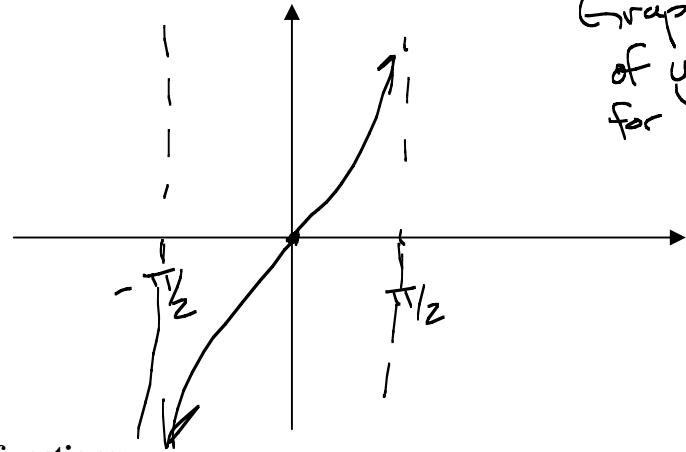
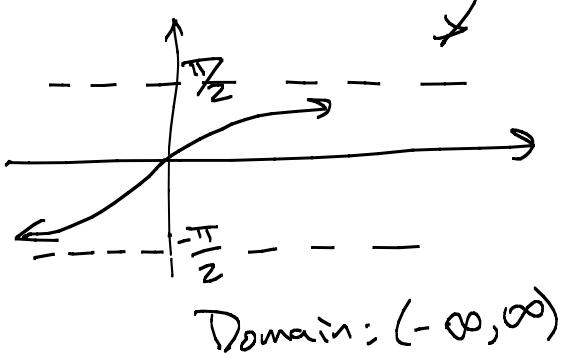
$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$$

Inverse tangent function:

$$y = \tan^{-1} x \quad \text{if and only if} \quad \tan y = x \quad \text{and} \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$ and $\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$

Graph of $y = \tan^{-1} x$:



Inverse secant, cosecant, and cotangent functions:

These are not used as often, and are not defined consistently. Our book defines them as follows:

$$y = \cot^{-1} x \quad \text{if and only if} \quad \cot y = x \quad \text{and} \quad y \in (0, \pi)$$

$$y = \csc^{-1} x \quad \text{if and only if} \quad \csc y = x \quad \text{and} \quad y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$$

$$y = \sec^{-1} x \quad \text{if and only if} \quad \sec y = x \quad \text{and} \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

An important identity:

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$

Differentiation of the inverse sine function:

Since $y = \sin x$ is continuous and differentiable, so is $y = \sin^{-1} x$.

We want to find its derivative.

$$y = \sin^{-1} x. \text{ Find } \frac{dy}{dx}.$$

$$x = \sin y \text{ and } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\text{Implicit diff: } \frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$

$$1 = \cos y \frac{dy}{dx}$$

$$\frac{1}{\cos y} = \frac{dy}{dx}$$

We need to write $\cos y$ in terms of x .

We know that $\cos^2 y + \sin^2 y = 1$ (Pythagorean trig identity)

$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = \pm \sqrt{1 - \sin^2 y}. \text{ We know } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \text{ so } \cos y > 0$$

$$\cos y > 0 \Rightarrow$$

$$\cos y = +\sqrt{1 - \sin^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

(Quadrants I, III)

$x = \sin y$ (from line 2 of proof)

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1}$$

Example 9: Differentiate $f(x) = \sin^{-1}(2x-7)$.

$$f'(x) = \frac{1}{\sqrt{1-(2x-7)^2}} \frac{d}{dx}(2x-7)$$

$$= \frac{1}{\sqrt{1-(2x-7)^2}} (2)$$

Differentiation of the inverse cosine function:

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

X
Very important

Differentiation of the inverse tangent function:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Derivatives of other inverse trigonometric functions:

X
Know these 3

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{ x \sqrt{x^2-1}}$
$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{ x \sqrt{x^2-1}}$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

no need to memorize these 3.

Example 10: Find the derivative of $f(x) = e^{\tan^{-1} 2x}$.

$$f(x) = e^{\tan^{-1}(2x)}$$

$$\begin{aligned} f'(x) &= e^{\tan^{-1}(2x)} \frac{d}{dx} (\tan^{-1}(2x)) \\ &= e^{\tan^{-1}(2x)} \cdot \frac{1}{1+(2x)^2} \frac{d}{dx} (2x) = \boxed{e^{\tan^{-1}(2x)} \frac{2}{1+4x^2}} \end{aligned}$$

Example 11: Find the derivative of $y = \csc^{-1}(\tan x)$.

Looking up The formula:

$$\frac{dy}{dx} = -\frac{1}{|\tan x| \sqrt{\tan^2 x - 1}} \frac{d}{dx} (\tan x)$$

$$= \boxed{-\frac{\sec^2 x}{|\tan x| \sqrt{\tan^2 x - 1}}}$$

$$\frac{d}{dx} (\csc^{-1} x) = -\frac{1}{|x| \sqrt{x^2 - 1}}$$

Example 12: Find the derivative of $f(x) = x^3 \arccos 2x$.

$$\begin{aligned} f'(x) &= x^3 \frac{d}{dx} (\arccos 2x) + (\arccos 2x) \frac{d}{dx} (x^3) \\ &= x^3 \left(-\frac{1}{\sqrt{1-x^2}} \right) + (\arccos 2x)(3x^2) \\ &= \boxed{-\frac{x^3}{\sqrt{1-x^2}} + 3x^2 \arccos 2x} \end{aligned}$$

Example 13: Find the equation of the line tangent to the graph of $f(x) = \arctan x$ at the point where $x = -1$.

$$f'(x) = \frac{1}{1+x^2}$$

$$\text{slope } m = f'(-1) = \frac{1}{1+(-1)^2} = \frac{1}{1+1} = \frac{1}{2}$$

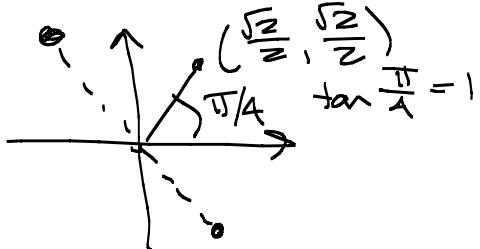
Find y-value: $f(-1) = \arctan(-1) = \tan^{-1}(-1) = -\frac{\pi}{4}$

Point: $(-1, -\frac{\pi}{4})$

$$y - y_1 = m(x - x_1)$$

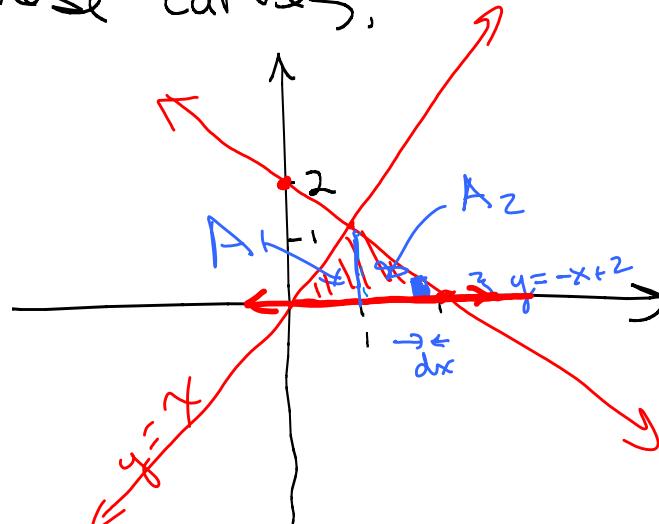
$$y - \left(-\frac{\pi}{4}\right) = \frac{1}{2}(x - (-1))$$

then clean it up.



Homework Questions
7.1 #21) Find area of region bounded by those curves,

$$\begin{cases} y=x \\ y=2-x \\ y=0 \\ y=-x+2 \end{cases} \text{ given}$$



$$A_1 = \int_0^1 x \, dx$$

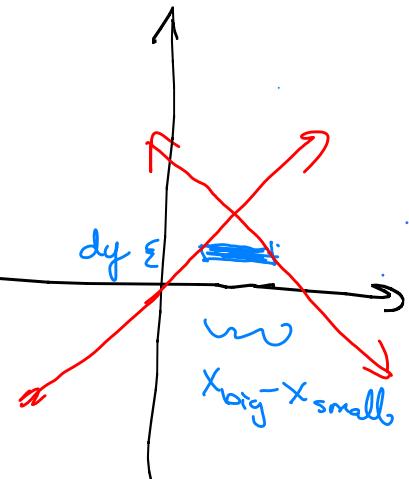
$$A_2 = \int_1^2 (2-x) \, dx$$

OR integrate with respect to y.

$$\text{Area} = \int_0^1 (x_{\text{big}} - x_{\text{small}}) \, dy = \int_0^1 (2-y - y) \, dy$$

$$x_{\text{big}}: y = -x + 2 \Rightarrow x_{\text{big}} = -y + 2 = 2 - y$$

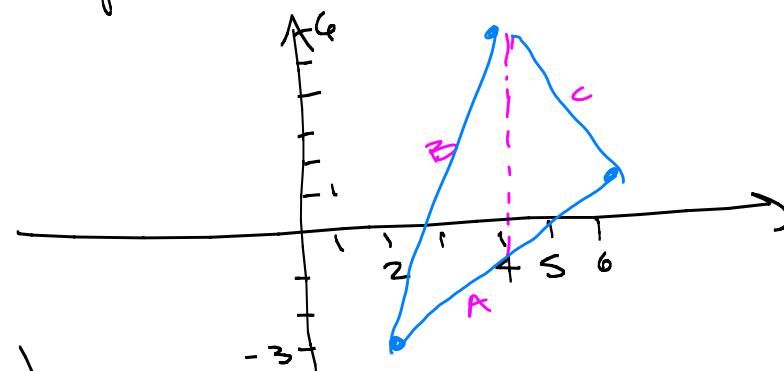
$$x_{\text{small}}: y = x \Rightarrow x_{\text{small}} = y$$



$$\# 55 \quad \begin{cases} (2, -3) \\ (4, 6) \\ (6, 1) \end{cases}$$

Find eqns of lines.
(Calculate m, then use
 $y - y_1 = m(x - x_1)$)

$$\text{Area} = \int_2^4 \left[\frac{9}{2}x - 12 - (x-5) \right] dx +$$



$$A: y = x - 5$$

$$B: y = \frac{9}{2}x - 12$$

$$C: y = -\frac{5}{2}x + 16$$