

$$\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

5.7.1

## 5.7: Inverse Trigonometric Functions – Integration

Two important integration rules come from the inverse trigonometric differentiation rules.

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C \quad \int \frac{1}{x^2+1} dx = \tan^{-1} x + C$$

Example 1: Find  $\int \frac{1}{\sqrt{1-9x^2}} dx$ .

$$\int \frac{1}{\sqrt{1-9x^2}} dx$$

$$= \int \frac{1}{\sqrt{1-(3x)^2}} dx$$

$$= \frac{1}{3} \int \frac{1}{\sqrt{1-u^2}} du = \frac{1}{3} \sin^{-1}(u) + C$$

$$\begin{aligned} u &= 3x \\ \frac{du}{dx} &= 3 \\ du &= 3dx \\ \frac{1}{3} du &= dx \end{aligned}$$

$$= \boxed{\frac{1}{3} \sin^{-1}(3x) + C}$$

Example 2: Find  $\int \frac{1}{x^2+7} dx$ .

$$\int \frac{1}{x^2+7} dx = \int \frac{1}{7(\frac{x^2}{7}+1)} dx = \frac{1}{7} \int \frac{1}{\frac{x^2}{7}+1} dx$$

$$= \frac{1}{7} \int \frac{1}{1+(\frac{x}{\sqrt{7}})^2} dx$$

$$= \frac{1}{7} \cdot \sqrt{7} \int \frac{1}{1+u^2} du$$

$$= \frac{\sqrt{7}}{7} \tan^{-1}(u) + C = \boxed{\frac{\sqrt{7}}{7} \tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + C}$$

$$\begin{aligned} u &= \frac{x}{\sqrt{7}} = \frac{1}{\sqrt{7}}x \\ \frac{du}{dx} &= \frac{1}{\sqrt{7}} \\ du &= \frac{1}{\sqrt{7}} dx \\ \sqrt{7} du &= dx \end{aligned}$$

More general forms of these integration rules are

$$\left\{ \begin{array}{l} \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C \\ \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left( \frac{x}{a} \right) + C \end{array} \right.$$

I will give these to you

now it looks like formula above, with  $a=5$

Example 3: Find  $\int \frac{4x}{x^4 + 25} dx$ .

$$\begin{aligned} \int \frac{4x}{(x^2)^2 + 25} dx &= 4 \left( \frac{1}{2} \right) \int \frac{1}{u^2 + 25} du \\ &= 2 \int \frac{1}{u^2 + 5^2} du \\ &= 2 \left( \frac{1}{5} \right) \tan^{-1} \left( \frac{u}{5} \right) + C \\ &= \boxed{\frac{2}{5} \tan^{-1} \left( \frac{x^2}{5} \right) + C} \end{aligned}$$

using  $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$ , with  $u=x$ ,  $a=5$

Another antiderivative:

$$\boxed{\int \frac{1}{x\sqrt{x^2 - 1}} dx = \sec^{-1}|x| + C}$$

Example 4: Find  $\int \frac{1}{x\sqrt{x^2 - 4}} dx$ .

$$\begin{aligned} \int \frac{1}{x\sqrt{x^2 - 4}} dx &= \int \frac{1}{x\sqrt{1 + (\frac{x^2}{4} - 1)}} dx = \int \frac{1}{x\sqrt{1 + \left(\frac{x^2}{4}\right)^2 - 1}} dx \\ &= \frac{1}{\sqrt{4}} \int \frac{1}{x\sqrt{\left(\frac{x^2}{2}\right)^2 - 1}} dx \\ &= \frac{1}{\sqrt{4}} \cdot 2 \int \frac{1}{2u\sqrt{u^2 - 1}} du \quad \text{pull this 2 out} \\ &= \frac{1}{2} \cdot 2 \cdot \frac{1}{2} \int \frac{1}{u\sqrt{u^2 - 1}} du = \frac{1}{2} \sec^{-1}|u| + C = \boxed{\frac{1}{2} \sec^{-1}\left|\frac{x}{2}\right| + C} \end{aligned}$$

want to get a formula.

I have this!

I need,

u =  $\frac{x}{2}$  =  $\frac{1}{2}x$   
 $\frac{du}{dx} = \frac{1}{2}$   
 $du = \frac{1}{2}dx$   
 $2du = dx$   
 $x = 2u$

want to  
make it look  
like  $\int \frac{1}{x^2+a^2} dx$

Complete the square:

5.7.3

Example 5: Find  $\int \frac{1}{4x^2 - 12x + 17} dx$ .

$$\int \frac{1}{4(x-\frac{3}{2})^2 + 8} dx$$

$$\int \frac{1}{4((x-\frac{3}{2})^2 + 2)} dx$$

$$= \frac{1}{4} \int \frac{1}{(x-\frac{3}{2})^2 + 2} dx$$

$$= \frac{1}{4} \int \frac{1}{(x-\frac{3}{2})^2 + (\sqrt{2})^2} dx = \frac{1}{4} \int \frac{1}{u^2 + (\sqrt{2})^2} du$$

Example 6: Find  $\int \frac{x-7}{\sqrt{5-4x^2}} dx$ .

$$\int \frac{x-7}{\sqrt{5-4x^2}} dx$$

$$= \int \frac{x}{\sqrt{5-4x^2}} dx - \int \frac{7}{\sqrt{5-4x^2}} dx$$

$$u = 5-4x^2$$

$$du = -8x dx$$

$$-\frac{1}{8} du = dx$$

$$= -\frac{1}{8} \int \frac{1}{\sqrt{u}} du - 7 \int \frac{1}{\sqrt{(5)^2 - (2x)^2}} dx$$

$$u = x - \frac{3}{2}$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$a = \sqrt{2}$$

$$= \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= \boxed{\frac{1}{4\sqrt{2}} \tan^{-1}\left(\frac{x-3/2}{\sqrt{2}}\right) + C}$$

$$u = 2x$$

$$du = 2dx$$

$$\frac{1}{2} du = dx$$

use formula

$$\int \frac{1}{\sqrt{a^2 - u^2}} du = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\text{with } \frac{x-u}{a} = \frac{x-2x}{\sqrt{5}} = \frac{-x}{\sqrt{5}}$$

$$= -\frac{1}{8} \cdot \frac{u^{1/2}}{1/2} - \frac{7}{2} \sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$$

$$= -\frac{1}{8} \cdot 2u^{1/2} - \frac{7}{2} \sin^{-1}\left(\frac{2x}{\sqrt{5}}\right) + C$$

$$= \boxed{-\frac{1}{4} \sqrt{5-4x^2} - \frac{7}{2} \sin^{-1}\left(\frac{2x}{\sqrt{5}}\right) + C}$$