

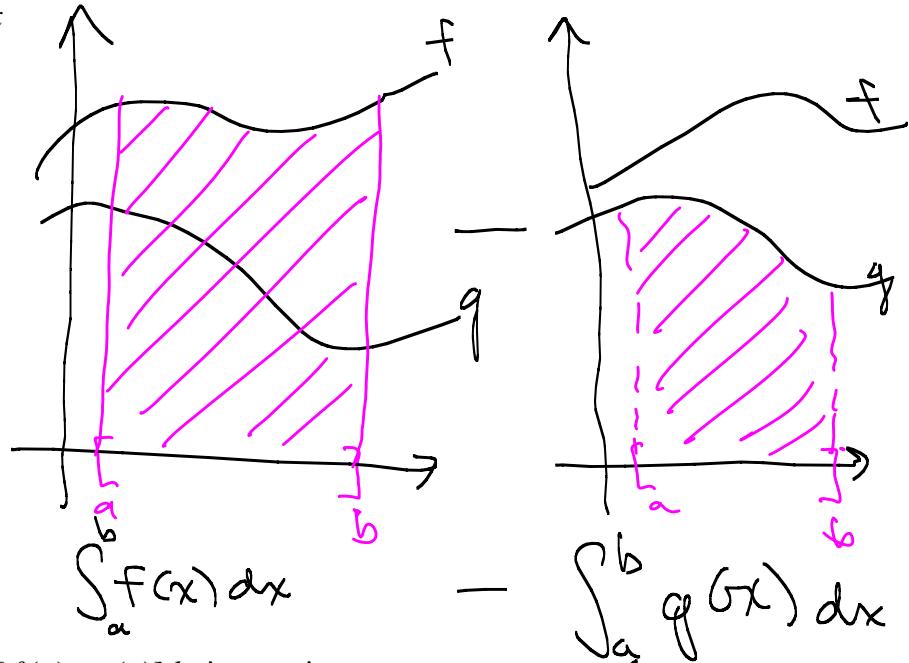
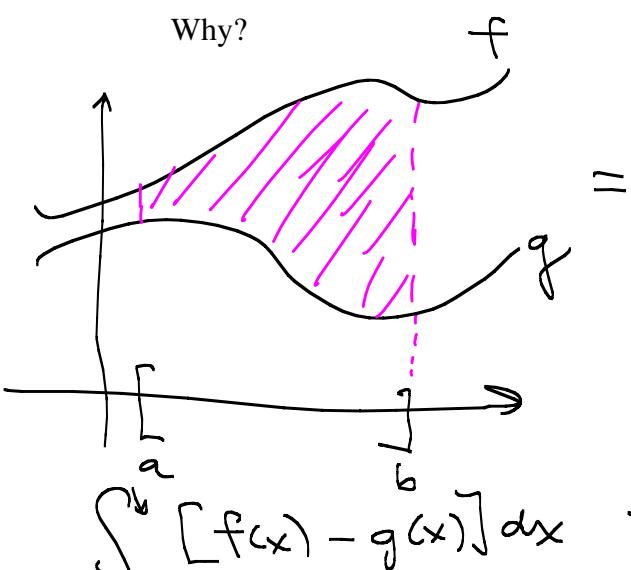
## 7.1: Area of a Region Between Two Curves

Because the definite integral represents the “net” area under a curve, we can use integration to find the area between curves.

If  $f$  and  $g$  are continuous and  $f(x) \geq g(x)$  on  $[a, b]$ , then the area between  $y = f(x)$ ,  $y = g(x)$ , and the lines  $x = a$  and  $x = b$  is given by

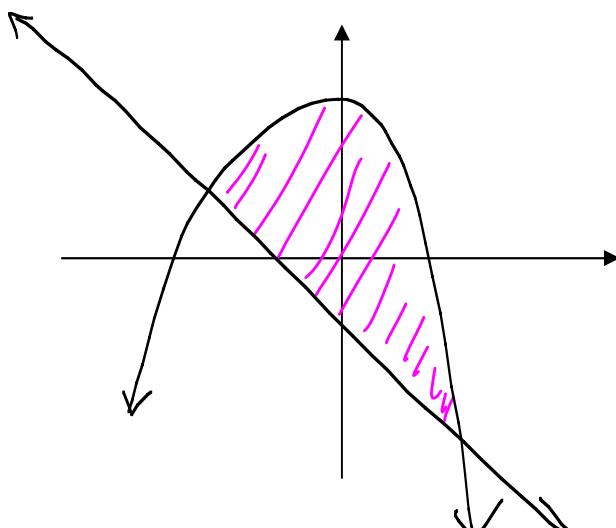
$$\text{Area} = \int_a^b [f(x) - g(x)] dx$$

Why?

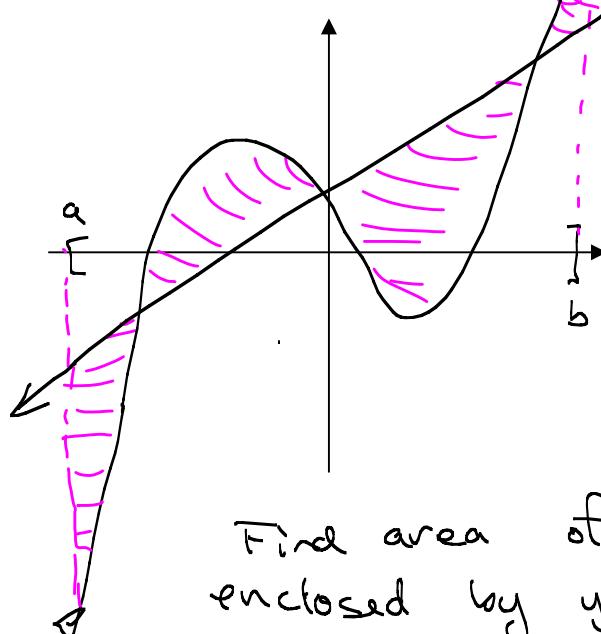


Note: If  $g(x) \geq f(x)$  on  $[a, b]$ , then  $\int_a^b [f(x) - g(x)] dx$  is negative.

Some different types of area scenarios:



Find area enclosed by  $y = f(x)$  and  $y = g(x)$ .



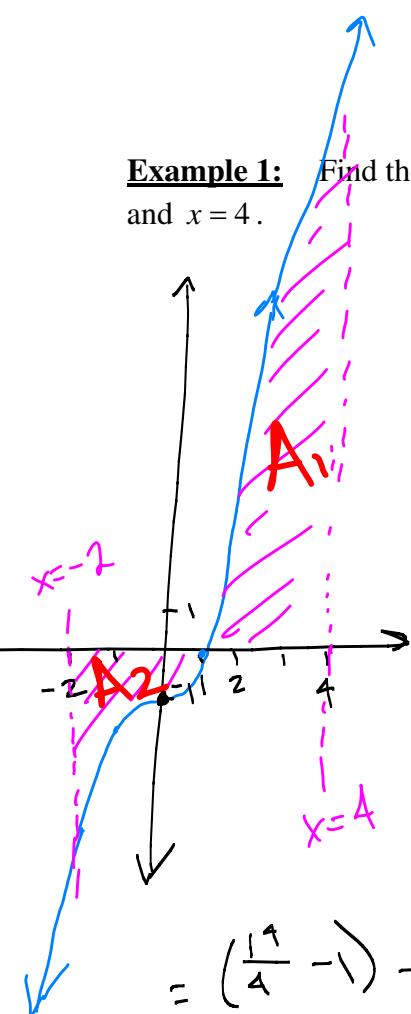
Find area of region enclosed by  $y = f(x)$ ,  $y = g(x)$ ,  $x = a$ ,  $x = b$ .

7.1.2

**Example 1:** Find the area of the region bounded by  $f(x) = x^3 - 1$  and the lines  $y = 0$ ,  $x = -2$ , and  $x = 4$ .

$\int_a^b (x^3 - 1) dx$  will give us <sup>net</sup> area between  $y = x^3 - 1$  and  $x$ -axis

Where does  $y = x^3 - 1$  intersect with  $y = 0$ ?  
set  $0 = x^3 - 1$ . Then  $1 = x^3$ , so  $x = 1$ .



$$A_1 = \int_{-2}^1 (x^3 - 1) dx = \left( \frac{x^4}{4} - x \right) \Big|_{-2}^1 = \left( \frac{1}{4} - 1 \right) - \left( \frac{(-2)^4}{4} - (-2) \right) = \frac{256}{4} - \frac{16}{4} - \frac{1}{4} + \frac{1}{4}$$

$$A_2 = \int_1^4 (x^3 - 1) dx = \left( \frac{x^4}{4} - x \right) \Big|_1^4 = \left( \frac{1}{4} - 1 \right) - \left( \frac{(-2)^4}{4} - (-2) \right) = \frac{1}{4} - \frac{1}{4} - \frac{16}{4} - \frac{8}{4} = -\frac{3}{4} - \frac{24}{4} = -\frac{27}{4}$$

$$\text{Area} = |A_1| + |A_2| = \left| \frac{243}{4} \right| + \left| -\frac{27}{4} \right| = \frac{243}{4} + \frac{27}{4} = \frac{270}{4} = \boxed{\frac{135}{2}}$$

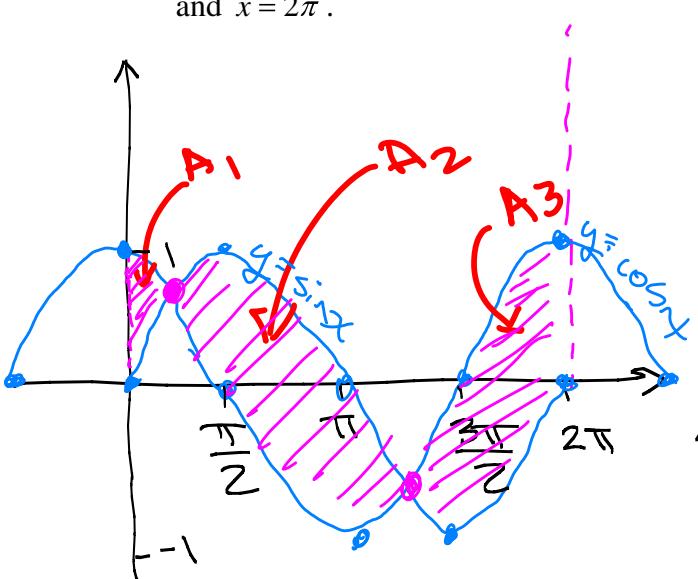
**Example 2:** Find the area of the region bounded by  $y = \cos x$ ,  $y = \sin x$ , and the lines  $x = 0$ , and  $x = 2\pi$ .

Find the intersection points:

Set y's equal:  $\cos x = \sin x$

$$\frac{\cos x}{\cos x} = \frac{\sin x}{\cos x} / \quad l = \tan x$$

$$1 = \tan x \quad x = \frac{\pi}{4}, \frac{5\pi}{4} \quad (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$$



$$A_1 = \int_0^{\pi/4} (\cos x - \sin x) dx = (\sin x - (-\cos x)) \Big|_0^{\pi/4} = (\sin x + \cos x) \Big|_0^{\pi/4} = \sin x \Big|_0^{\pi/4} + \cos x \Big|_0^{\pi/4}$$

$$= \sin \frac{\pi}{4} - \sin 0 + \cos \frac{\pi}{4} - \cos 0 = \frac{\sqrt{2}}{2} - 0 + \frac{\sqrt{2}}{2} - 1 = \frac{2\sqrt{2}}{2} - 1 = \sqrt{2} - 1$$

$$A_2 = \int_{\pi/4}^{5\pi/4} (\cos x - \sin x) dx = (\sin x + \cos x) \Big|_{\pi/4}^{5\pi/4}$$

(should be negative)

$$= \sin x \Big|_{\pi/4}^{5\pi/4} + \cos x \Big|_{\pi/4}^{5\pi/4}$$

$$= \sin\left(\frac{5\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)$$

$$= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}$$

$$= -\frac{2\sqrt{2}}{2} - \frac{2\sqrt{2}}{2} = -\sqrt{2} - \sqrt{2} = -2\sqrt{2}$$

$$A_3 = \int_{5\pi/4}^{2\pi} (\cos x - \sin x) dx = \sin x + \cos x \Big|_{5\pi/4}^{2\pi}$$

$$= (\sin 2\pi + \cos 2\pi) - \left(\sin \frac{5\pi}{4} + \cos \frac{5\pi}{4}\right)$$

$$= 0 + 1 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

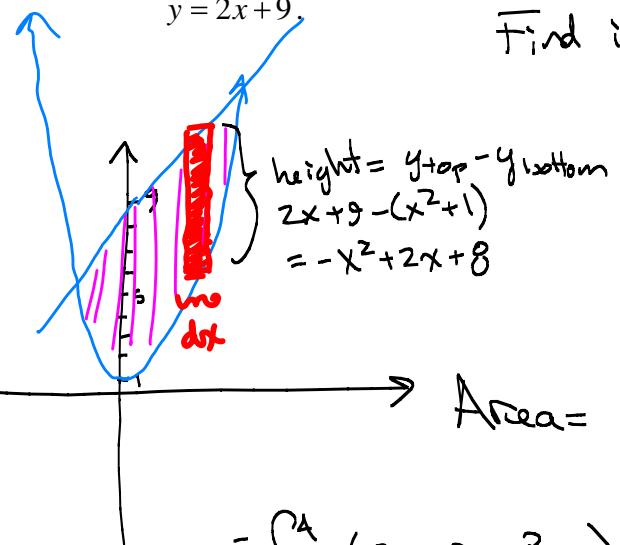
$$= 1 - \left(-\frac{2\sqrt{2}}{2}\right) = 1 + \frac{2\sqrt{2}}{2} = 1 + \sqrt{2}$$

$$\text{Area} = |A_1| + |A_2| + |A_3| = |\sqrt{2} - 1| + |-2\sqrt{2}| + |1 + \sqrt{2}|$$

$$= \sqrt{2} - 1 + 2\sqrt{2} + 1 + \sqrt{2}$$

$$= \boxed{4\sqrt{2}}$$

**Example 3:** Find the area of the region completely enclosed by the graphs of  $y = x^2 + 1$  and  $y = 2x + 9$



Find intersection pts: Set y's equal:

$$x^2 + 1 = 2x + 9$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

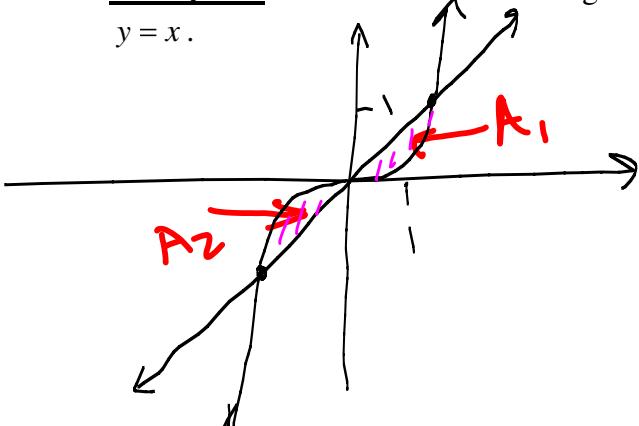
$x = 4, x = -2$  Seems reasonable from graph.

$$\text{Area} = \int_{-2}^4 (2x + 9 - (x^2 + 1)) dx$$

$$= \int_{-2}^4 (2x + 9 - x^2 - 1) dx = \int_{-2}^4 (-x^2 + 2x + 8) dx$$

$$= -\frac{x^3}{3} \Big|_{-2}^4 + \frac{2x^2}{2} \Big|_{-2}^4 + 8x \Big|_{-2}^4 = -\frac{4^3}{3} - \left(-\frac{(-2)^3}{3}\right) + 4^2 - (-2)^2 + 8(4) - 8(-2)$$

**Example 4:** Find the area of the region completely enclosed by the graphs of  $y = x^3$  and  $y = x$ .



$$\begin{aligned} &= -\frac{4}{3} + \frac{-8}{3} + 16 - 4 + 32 + 16 \\ &= -\frac{72}{3} + 12 + 48 = -24 + 60 = \boxed{36} \end{aligned}$$

Find intersection pts: Set y's equal:

$$x^3 = x$$

$$x^3 - x = 0$$

$$x(x^2 - 1) = 0$$

$$x(x+1)(x-1) = 0$$

$$x = 0, 1, -1$$

$$\text{Area} = A_1 + A_2 = 2A_1$$

$$= 2 \int_0^1 (x - x^3) dx$$

$$= 2 \left( \frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1$$

$$= \left( x^2 - \frac{x^4}{2} \right) \Big|_0^1 = \left( 1^2 - \frac{1^4}{2} \right) - \left( 0^2 - \frac{0^4}{2} \right) = 1 - \frac{1}{2}$$

$$= \boxed{\frac{1}{2}}$$

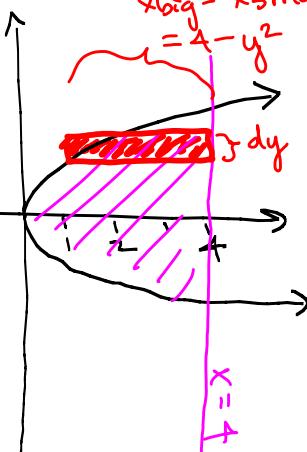
**Example 5:** Find the area of the region completely enclosed by the graphs of  $x = y^2$  and

$$x = 4. \quad x_{\text{big}} - x_{\text{small}} = 4 - y^2$$

Find intersection pts: set x's equal:

$$y^2 = 4$$

$$y = \pm 2$$



$$\text{Area} \approx \sum \text{height}(\text{width})$$

$$= \sum (4 - y^2) dy$$

$$\text{Area} = \int_{-2}^2 (4 - y^2) dy = 2 \int_0^2 (4 - y^2) dy$$

$$= 2 \left( 4y - \frac{y^3}{3} \right) \Big|_0^2 = \left( 8y - \frac{2y^3}{3} \right) \Big|_0^2 = \left( 8(2) - \frac{2(2)^3}{3} \right) - \left( 8(0) - \frac{2(0)^3}{3} \right)$$

$$= 16 - \frac{16}{3} = \frac{48}{3} - \frac{16}{3} = \boxed{\frac{32}{3}}$$

**Example 6:** Find the area of the region completely enclosed by the graphs of  $x = 3 - y^2$  and

$$x = y + 1.$$

$$x = y + 1$$

$$x - 1 = y$$

$$y = (x - 1)$$

$$\text{slope: } m = 1$$

$$y\text{-int: } b = -1$$

$$A = \int_{-2}^1 ((3 - y^2) - (y + 1)) dy$$

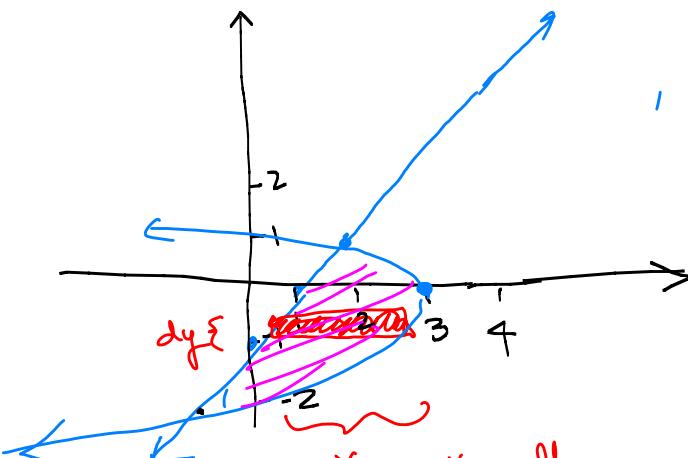
$$= \int_{-2}^1 (-y^2 - y + 2) dy$$

$$= -\frac{y^3}{3} \Big|_{-2}^1 - \frac{y^2}{2} \Big|_{-2}^1 + 2y \Big|_{-2}^1$$

$$= -\frac{1}{3} + \frac{(-2)^3}{3} - \frac{1}{2} + \frac{(-2)^2}{2} + 2(1) - 2(-2)$$

$$= -\frac{1}{3} - \frac{8}{3} - \frac{1}{2} + \frac{4}{2} + 2 + 4 = -\frac{2}{3} + \frac{3}{2} + 6 = -3 + \frac{3}{2} + 6$$

$$= \frac{6}{2} + \frac{3}{2} = \boxed{\frac{9}{2}}$$



$$x = -y^2 + 3$$

Find intersections:  
set x's equal  
 $3 - y^2 = y + 1$

$$0 = y^2 + y - 2$$

$$0 = (y+2)(y-1)$$

$$y = -2, y = 1$$

$$x_{\text{big}} - x_{\text{small}}$$

$$= 3 - y^2 - (y + 1)$$

$$= 3 - y^2 - y - 1 = -y^2 - y + 2$$