4.1: Antiderivatives and Indefinite Integration

Definition: An *antiderivative* of *f* is a function whose derivative is *f*.

i.e. A function *F* is an antiderivative of *f* if $F'(x) = f(x)$.

Example 1: $F(x) = x^3 + 5x$ is an antiderivative of $f(x) = 3x^2 + 5$.

check: $F'(x) = 3x^2 + 5$ ✓ok

What are some more antiderivatives of $f(x) = 3x^2 + 5$?

$G(x) = x^3 + 5x + 6$

$H(x) = x^3 + 5x - \frac{17\pi}{2}$

$J(x) = x^3 + 5x + C$, where C is any constant

So we have a whole "family" of antiderivatives of *f*.

> Definition: A function *F* is called an antiderivative of *f* on an interval *I* if $F'(x) = f(x)$ for all *x* in *I*.

> Theorem: If *F* is an antiderivative of *f* on an interval *I*, then all antiderivatives of *f* on *I* will be of the form
>
> $F(x) + C$
>
> where *C* is an arbitrary constant.

Example 2: Find the general form of the antiderivatives of $f(x) = 3x^2 + 5$.

$F(x) = x^3 + 5x + c$

Example 3: Find the general form of the antiderivatives of $f(x) = 6x^5 + \cos x$.

$F(x) = x^6 + \sin x + c$

Check: $\frac{d}{dx}(x^6 + \sin x + c)$

$= 6x^5 + \cos x + 0 = f(x)$ ✓

Integration:

<u>Integration</u> is the process of finding antiderivatives.

$\int f(x)dx \rightarrow$

> $\int f(x)dx$ is called the *indefinite integral* of *f*.
>
> $\int f(x)dx$ is the family of antiderivatives, or the most general antiderivative of *f*.
>
> This means: $\int f(x)dx = F(x) + c$, where $F'(x) = f(x)$.
>
> The *c* is called the *constant of integration*.

Integral sign

"with respect to *x*"

$$\int f(x)dx$$

Integrand

<u>Example 4:</u> Find $\int 3x^2 + 5\, dx$.

$$\int (3x^2 + 5)\, dx = \boxed{x^3 + 5x + c}$$

<u>Example 5:</u> Find $\int 6x^5 + \cos x\, dx$.

$$\int (6x^5 + \cos x)\, dx = \boxed{x^6 + \sin x + c}$$

<u>Example 6:</u> Find $\int \sec^2 x\, dx$.

$$\int \sec^2 x\, dx = \boxed{\tan x + C}$$

<u>Check:</u> $\frac{d}{dx}(\tan x + c) = \sec^2 x$ ✓

Rules for Finding Antiderivatives:

Notation in this table: F is an antiderivative of f, G is an antiderivative of g,

Function	Antiderivative
k	$kx+c$
$kf(x)$	$kF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
x^n for $n \neq -1$	$\dfrac{x^{n+1}}{n+1}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\csc^2 x$	$-\cot x$
$\csc x \cot x$	$-\csc x$

Note: $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right)$

$= \frac{1}{\cancel{n+1}}(\cancel{(n+1)})x^n = x^n$

Check:

$\frac{d}{dx}(-\cot x) = -(-\csc^2 x)$

$= \csc^2 x$ ✓

$\frac{d}{dx}(-\csc x) = -(-\csc x \cot x)$

$= \csc x \cot x$ ✓

1. $\int k \ dx = kx + c \quad$ (k a constant)

2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c \quad (n \neq -1)$

3. $\int k \ f(x) \ dx = k\int f(x) \ dx \quad$ (k a constant)

4. $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

5. $\int \cos x \, dx = \sin x + c$

6. $\int \sin x \, dx = -\cos x + c$

7. $\int \sec^2 x \, dx = \tan x + c$

8. $\int \sec x \tan x \, dx = \sec x + c$

9) $\int \csc^2 x \, dx = -\cot x + c$

10) $\int \csc x \cot x \, dx = -\csc x + c$

Example 7: Find the general antiderivative of $f(x) = \dfrac{1}{2}$.

$F(x) = \frac{1}{2}x + c$

Check: $F'(x) = \frac{1}{2}$ ✓

Use $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ (Power Rule) $(n \neq -1)$

Example 8: Find $\int x^3\,dx$.

$$\int x^3 dx = \frac{x^{3+1}}{3+1} + C = \boxed{\frac{x^4}{4} + C}$$

Check: $\frac{d}{dx}\left(\frac{x^4}{4}\right) = \frac{d}{dx}\left(\frac{1}{4}x^4\right) = \frac{1}{4}(4x^3) = x^3$ ✓

Example 9: Find $\int 7x^2\,dx$.

$$\int 7x^2 dx = 7\int x^2 dx = 7 \cdot \frac{x^{2+1}}{2+1} + C = \boxed{\frac{7x^3}{3} + C}$$

Check: $\frac{d}{dx}\left(\frac{7}{3}\cdot x^3\right) = \frac{7}{3}(3x^2) = 7x^2$ ✓

Example 10: Find $\int \frac{1}{x^5}\,dx$.

$$\int \frac{1}{x^5} dx = \int x^{-5} dx = \frac{x^{-5+1}}{-5+1} + C = \frac{x^{-4}}{-4} + C = \boxed{-\frac{1}{4x^4} + C}$$

Check: $\frac{d}{dx}\left(-\frac{1}{4}\cdot x^{-4}\right) = -\frac{1}{4}(-4x^{-5}) = x^{-5}$ ✓ ok

Example 11: Find the general antiderivative of $f(x) = \frac{5}{x^2}$.

Example 12: $\int \left(6x^2 - 3x + 9\right) dx$

Example 13: Find $\int 3\sqrt{x}\,dx$.

Example 14: Find $\int (3\cos x + 5\sin x)\, dx$.

Example 15: Find $\int \frac{x^7 - \sqrt[3]{x} + 3x^2}{x^5}\, dx$.

Example 16: Find the general antiderivative of $f(\theta) = \frac{\sin\theta}{3}$.

$$F(\theta) = \int \frac{\sin\theta}{3}\, d\theta = \frac{1}{3}\int \sin\theta\, d\theta = \frac{1}{3}(-\cos\theta) + c$$

$$= \boxed{-\frac{1}{3}\cos\theta + c}$$

$$= \boxed{-\frac{\cos\theta}{3} + c}$$

Example 17: $\int \left(\sqrt[3]{x} + \frac{2}{\sqrt{x}} \right) dx$

$$\int \left(x^{\frac{1}{3}} + 2x^{-\frac{1}{2}}\right) dx$$

$$= \frac{x^{4/3}}{4/3} + 2\frac{x^{1/2}}{1/2} + c = \boxed{\frac{3x^{4/3}}{4} + 4x^{1/2} + c}$$

Example 18: $\int (6y^2 - 2)(8y + 5)\, dy$

$$\int (48y^3 + 30y^2 - 16y - 10)\, dy$$

$$= \frac{48y^4}{4} + \frac{30y^3}{3} - \frac{16y^2}{2} - 10y + c$$

$$= \boxed{12y^4 + 10y^3 - 8y^2 - 10y + c}$$

check:

$$\frac{d}{dy}(12y^4 + 10y^3 - 8y^2 - 10y)$$

$$= 48y^3 + 30y^2 - 16y - 10 \checkmark$$

Differential equations:

A *differential equation* is an equation involving the derivative of a function. To solve a differential equation means to find the original function.

An *initial value problem* is a common type of differential equation in which a derivative and an initial condition are given.

Example 19: Given $f'(x) = x^2 - 7$, find *f*. This is an example of a differential equation.

$$f(x) = \int f'(x)\,dx = \int (x^2 - 7)\,dx = \frac{x^3}{3} - 7x + C$$

Solution: $\boxed{f(x) = \frac{x^3}{3} - 7x + C}$ ← general antiderivative

Check: $f'(x) = \frac{1}{3}(3x^2) - 7$
$= x^2 - 7$ ✓

Example 20: Suppose that $f'(x) = 3x^2 + 2\cos x$ and $f(0) = 3$. Find $f(x)$.

$$f(x) = \int f'(x)\,dx = \int (3x^2 + 2\cos x)\,dx$$
$$= \frac{3x^3}{3} + 2\sin x + C$$
$$= x^3 + 2\sin x + C$$

Find c: $f(0) = 0^3 + 2\sin 0 + C = 3$
$0 + 0 + C = 3$
$C = 3$

Solution: $\boxed{f(x) = x^3 + 2\sin x + 3}$

Example 21: Suppose that $f''(x) = 2x^3 - 6x^2 + 6x$, $f'(2) = -1$, and $f(-1) = 4$. Find $f(x)$.

$$f'(x) = \int f''(x)\,dx = \int (2x^3 - 6x^2 + 6x)\,dx = \frac{2x^4}{4} - \frac{6x^3}{3} + \frac{6x^2}{2} + c_1$$

$$= \frac{x^4}{2} - 2x^3 + 3x^2 + c_1$$

Find c_1: $f'(2) = -1 \Rightarrow f'(2) = \frac{2^4}{2} - 2(2)^3 + 3(2)^2 + c_1 = -1$

$$8 - 16 + 12 + c_1 = -1$$

$$4 + c_1 = -1$$

$$c_1 = -5$$

$$f'(x) = \frac{x^4}{2} - 2x^3 + 3x^2 - 5$$

$$f(x) = \int f'(x)\,dx = \int \left(\frac{x^4}{2} - 2x^3 + 3x^2 - 5\right)dx$$

$$= \frac{1}{2}\cdot\frac{x^5}{5} - \frac{2x^4}{4} + 3\frac{x^3}{3} - 5x + c_2$$

$$= \frac{1}{10}x^5 - \frac{x^4}{2} + x^3 - 5x + c_2$$

$f(-1) = 4 \Rightarrow f(-1) = \frac{1}{10}(-1)^5 - \frac{(-1)^4}{2} + (-1)^3 - 5(-1) + c_2 = 4$

$\boxed{\text{cont'd next page}}$

Example 22: Suppose that $f''(x) = 12x^2 - 18x$, $f(1) = 2$, and $f(-3) = 1$. Find $f(x)$.

$$f'(x) = \int f''(x)\,dx = \int (12x^2 - 18x)\,dx = 12\frac{x^3}{3} - 18\frac{x^2}{2} + c_1$$

$$= 4x^3 - 9x^2 + c_1$$

$$f(x) = \int f'(x)\,dx = \int (4x^3 - 9x^2 + c_1)\,dx = \frac{4x^4}{4} - 9\frac{x^3}{3} + c_1x + c_2$$

$$= x^4 - 3x^3 + c_1x + c_2$$

$f(1) = 2 \Rightarrow f(1) = 1^4 - 3(1)^3 + c_1(1) + c_2 = 2$

$$1 - 3 + c_1 + c_2 = 2$$

$$-2 + c_1 + c_2 = 2$$

$$c_1 + c_2 = 4$$

$f(-3) = 1 \Rightarrow f(-3) = (-3)^4 - 3(-3)^3 + c_1(-3) + c_2 = 1$

$$81 + 81 - 3c_1 + c_2 = 1$$

$$162 - 3c_1 + c_2 = 1$$

$$-3c_1 + c_2 = -161$$

System of 2 equations in 2 unknowns $\begin{cases} -3c_1 + c_2 = -161 \\ c_1 + c_2 = 4 \end{cases}$

cont'd next page

Ex 21 cont'd

$$f(-1) = \frac{1}{10}(-1)^5 - \frac{(-1)^4}{2} + (-1)^3 - 5(-1) + c_2 = 4$$

$$-\frac{1}{10} - \frac{1}{2} - 1 + 5 + c_2 = 4$$

$$-\frac{1}{10} - \frac{5}{10} + \cancel{4} + c_2 = \cancel{4}$$

$$-\frac{6}{10} + c_2 = 0$$

$$c_2 = \frac{6}{10} = \frac{3}{5}$$

$$\boxed{f(x) = \frac{1}{10}x^5 - \frac{1}{2}x^4 + x^3 - 5x + \frac{3}{5}}$$

Ex 22 cont'd:

Solve the system: $\begin{cases} -3c_1 + c_2 = -161 \\ c_1 + c_2 = 4 \end{cases}$

Subtract: $-4c_1 = -165$

$$c_1 = \frac{-165}{-4} = \frac{165}{4}$$

$$c_1 + c_2 = 4 \implies \frac{165}{4} + c_2 = 4$$

$$c_2 = \frac{16}{4} - \frac{165}{4}$$

$$c_2 = -\frac{149}{4}$$

Solution: $\boxed{f(x) = x^4 - 3x^3 + \frac{165}{4}x - \frac{149}{4}}$ Check it!

$$f'(x) = 4x^3 - 9x^2 + \frac{165}{4}$$

$$f''(x) = 12x^2 - 18x \quad \checkmark$$

$$f(1) = 1^4 - 3(1)^3 + \frac{165}{4}(1) - \frac{149}{4}$$

$$= 1 - 3 + \frac{16}{4} = -2 + 4 = 2 \checkmark$$

$$f(-3) = (-3)^4 - 3(-3)^3 + \frac{165}{4}(-3) - \frac{149}{4}$$

$$= 81 + 81 - \frac{495}{4} - \frac{149}{4}$$

$$= 162 - \frac{644}{4} = 162 - 161 = 1$$

$$\begin{array}{r} 165 \\ \times\ 3 \\ \hline 495 \\ 149 \\ \hline 644 \end{array} \qquad \begin{array}{r} 161 \\ 4\overline{)644} \\ 4 \\ \hline 24 \\ 24 \\ \hline 04 \end{array}$$

Velocity and acceleration (rectilinear motion):

We already know that if $f(t)$ is the position of an object at time *t*, then $f'(t)$ is its velocity and $f''(t)$ is its acceleration.

<u>Note</u>: Acceleration due to gravity near the earth's surface is approximately 9.8 m/s^2 or 32 ft/s^2.

<u>Example 23:</u> Suppose a particle's velocity is given by $v(t) = 2\sin t + \cos t$ and its initial position is $s(0) = 3$. Find the position function of the particle.

$s'(t) =$

$$s(t) = \int v(t)\,dt = \int s'(t)\,dt = \int (2\sin t + \cos t)\,dt$$

$$= -2\cos t + \sin t + c$$

$$s(0) = 3 \Rightarrow s(0) = -2\cos(0) + \sin(0) + c = 3$$

$$-2(1) + 0 + c = 3$$

$$-2 + c = 3$$

$$c = 5$$

Position function: $\boxed{s(t) = -2\cos t + \sin t + 5}$

$\uparrow +$ water

<u>Example 24:</u> Suppose a ball is thrown upward from a 30-foot bridge over a river at an initial velocity of 40 feet per second. How high does it go? When does it hit the water?

acceleration: $a(t) = -32$

$$s''(t) = -32$$

Velocity: $v(t) = s'(t) = \int s''(t)\,dt = \int (-32)\,dt = -32t + c_1$

Find c_1: $v(0) = 40 \Rightarrow v(0) = -32(0) + c_1 = 40$

$$c_1 = 40$$

So $v(t) = -32t + 40$

Position: $s(t) = \int v(t)\,dt = \int s'(t)\,dt = \int (-32t + 40)\,dt$

$$= -32\frac{t^2}{2} + 40t + c_2 = -16t^2 + 40t + c_2$$

$$s(0) = 30 \Rightarrow s(0) = -16(0)^2 + 40(0) + c_2 = 30$$

$$0 + 0 + c_2 = 30$$

$$c_2 = 30$$

Position function: $\boxed{s(t) = -16t^2 + 40t + 30}$

cont'd next page

Note: For an object acted upon by gravity (g) only, with an intial velocity of v_0 and an initial position of s_0, the position function is:

$$s(t) = -\frac{1}{2} g t^2 + v_0 t + h_0$$

How high does it go? Set $v = 0$:

$$v(t) = -32t + 40 = 0$$

$$-32t = -40$$

$$t = \frac{-40}{-32} = \frac{5}{4} = 1.25 \text{ seconds}$$

$$s(1.25) = -16(1.25)^2 + 40(1.25) + 30$$

$$= \boxed{55 \text{ ft}}$$

When does it hit the water?

Set $s(t) = 0$:

$$0 = -16t^2 + 40t + 30$$

Quadratic Formula: $t = \dfrac{-40 \pm \sqrt{40^2 - 4(-16)(30)}}{2(-16)}$

$$= \frac{20 \pm \sqrt{880}}{16} \approx 3.104 \text{ sec}, \cancel{-0.604 \text{ sec}}$$

throw out

It hits the water at $\boxed{3.104 \text{ seconds}}$