4.2: Area

There are two fundamental problems addressed by calculus:

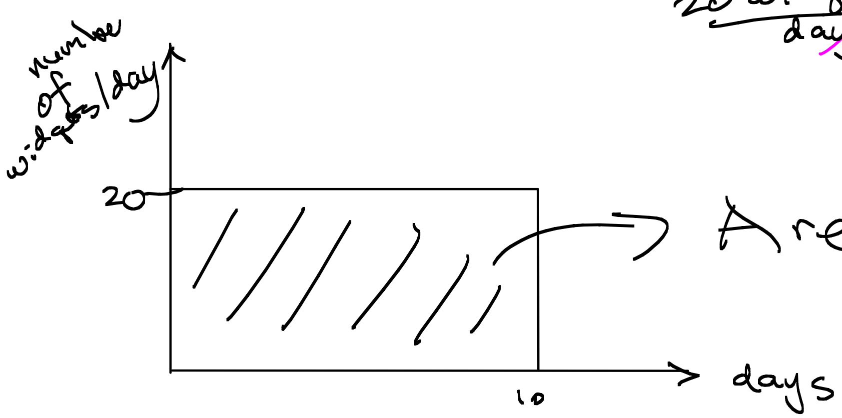
1. Finding the slope of a tangent line to a function at a particular location (differential calculus).
2. Finding the area between the graph of a function and the *x*-axis over an interval (integral calculus).

So far, we have spent most of our time learning differential calculus. Now we move on to integral calculus, which wouldn't be possible without the stuff we learned about differential calculus (derivatives, etc.).

Why do we care about area under the curve? Here's the idea…

Example 1: Suppose that a factory produces exactly 20 widgets per day. How many total widgets are produced in 10 days?

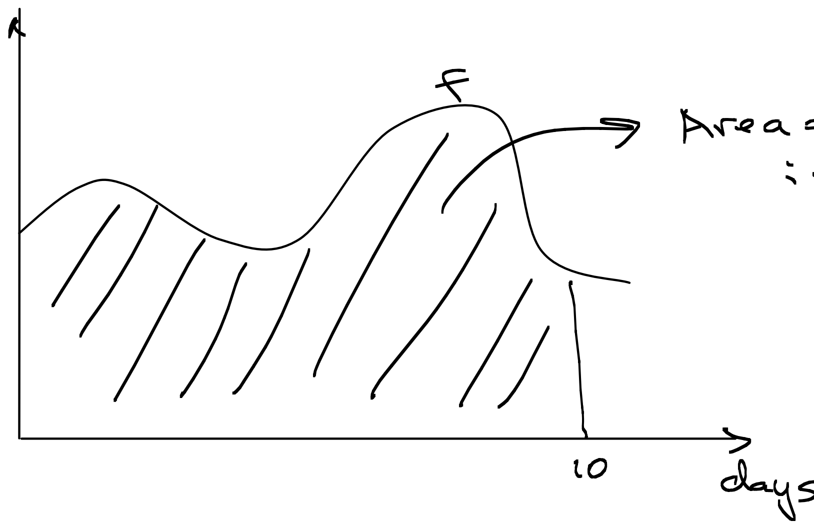
$\frac{20 \text{ widgets}}{\cancel{\text{day}}}(10\ \cancel{\text{days}}) = 200$ widgets total



Area = 20(10) = 200

Now suppose the number of widgets produced is given by the graph below. How many widgets are produced in days 1-10?

$f(x) =$ number of widgets per day at time x



Area = total # of widgets in 10 days

So, "total units produced" is equivalent to "area under the curve".

Review of Sigma notation:

Greek letter capital sigma: stands for Sum

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$$

index variable

Example 2: Evaluate $\sum_{k=1}^{5} \frac{k}{k+2}$.

$$= \underbrace{\frac{1}{1+2}}_{k=1} + \underbrace{\frac{2}{2+2}}_{k=2} + \underbrace{\frac{3}{3+2}}_{k=3} + \underbrace{\frac{4}{4+2}}_{k=4} + \underbrace{\frac{5}{5+2}}_{k=5}$$

$$= \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} = \boxed{\frac{197}{70}}$$

Example 3: Evaluate $\sum_{k=0}^{5} (-1)^k 2^k$

$$= \underbrace{(-1)^0 2^0}_{k=0} + \underbrace{(-1)^1 2^1}_{k=1} + \underbrace{(-1)^2 2^2}_{k=2} + \underbrace{(-1)^3 2^3}_{k=3} + \underbrace{(-1)^4 2^4}_{k=4} + \underbrace{(-1)^5 2^5}_{k=5}$$

$$= 2^0 - 2^1 + 2^2 - 2^3 + 2^4 - 2^5$$

$$= 1 - 2 + 4 - 8 + 16 - 32 = -1 - 4 - 16 = \boxed{-21}$$

Useful summation formulas:

Note: You do not need to memorize or prove these! They will come in handy when we use limits to evaluate the area under a curve.

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \quad \text{(where } c \text{ is a constant)} \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$$

Add the numbers from 1 to 100: $\sum_{i=1}^{100} i = \frac{100(101)}{2} = \frac{10100}{2} = 5050$

Area under the curve:

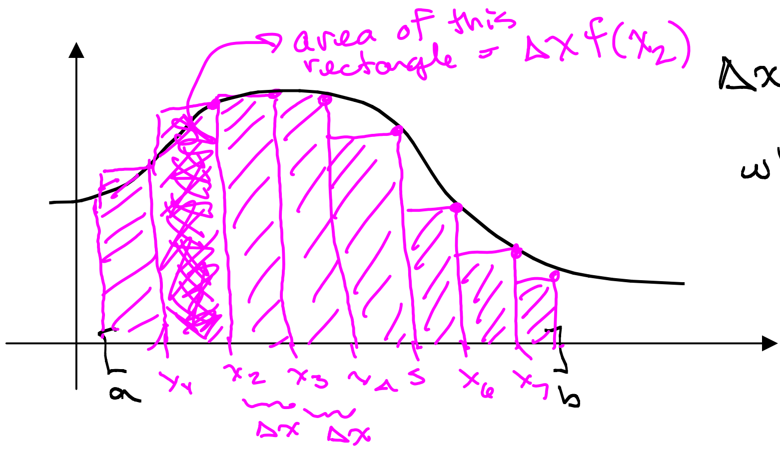
To find the area under the curve, we can approximate it using rectangles.

For equal width rectangles,

$$\Delta x = \frac{b-a}{n},$$

where n = the number of rectangles.

Example 4:



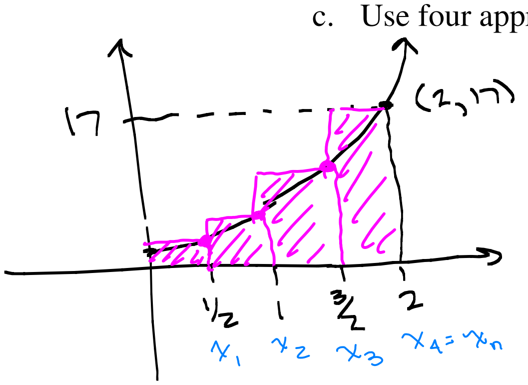
Δx means "change in x". This is the width of each rectangle.

The $f(x_i)$'s are the heights of the rectangles. They can be measured at the left endpoints, midpoints or right endpoints of the subintervals. (I used right endpoints in the example above)

The sum $A \approx f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$ is an example of a *Riemann sum*.

Example 5: Approximate the area under the graph of $f(x) = 4x^2 + 1$ over the interval [0,2].

a. Use four approximating rectangles and right endpoints.
b. Use four approximating rectangles and left endpoints.
c. Use four approximating rectangles and midpoints.



$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2}$$

$n = 4$

(a) Using Right Endpoints

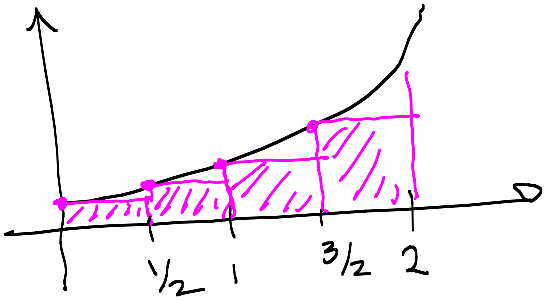
$$R_4 = \sum_{i=1}^{4} f(x_i)\Delta x$$

$$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$= \Delta x \left(f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2)\right)$$

$$= \tfrac{1}{2}(34) = \boxed{17}$$

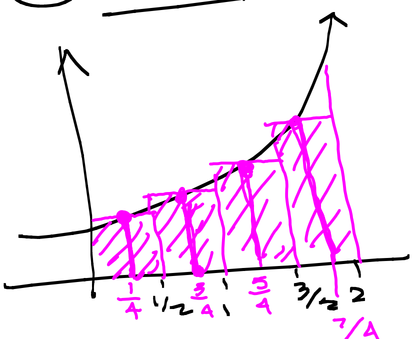
(b) Left endpoints



Area:

$L_4 = \Delta x f(0) + \Delta x f(\tfrac{1}{2})$
$\quad + \Delta x f(1) + \Delta x f(\tfrac{3}{2})$
$= \Delta x \left(f(0) + f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2})\right)$
$= \tfrac{1}{2}(1 + 2 + 5 + 10) = \tfrac{1}{2}(18)$
$= \boxed{9}$

(c) Midpoints



$M_4 = \Delta x f(\tfrac{1}{4}) + \Delta x f(\tfrac{3}{4})$
$\quad + \Delta x f(\tfrac{5}{4}) + \Delta x f(\tfrac{7}{4})$
$= \Delta x \left(f(\tfrac{1}{4}) + f(\tfrac{3}{4}) + f(\tfrac{5}{4}) + f(\tfrac{7}{4})\right)$
$= \tfrac{1}{2}(1.25 + 3.25 + 7.25 + 13.25)$
$= \tfrac{1}{2}(25) = \boxed{12.5}$

<u>Limit Definition for Area</u>: Let f be continuous and nonnegative on the interval $[a,b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x ,$$

where $x_{i-1} \le c_i \le x_i$ and $\Delta x = \dfrac{b-a}{n}$.

In other words, the area A of the region that lies under the graph of the nonnegative continuous function f is the limit of the sum of the areas of approximating rectangles. The height of each rectangle is found be evaluating the function at any c_i within the subinterval covered by that rectangle.

$$A = \lim_{n\to\infty} [f(c_0)\Delta x + f(c_1)\Delta x + \ldots + f(c_{n-1})\Delta x]$$

Upper and lower sums:

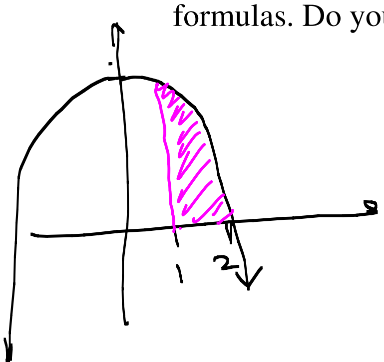
<u>Upper sum</u>: The sum of areas of approximating rectangles, with the height of each rectangle equal to the *maximum* y-value on that subinterval.

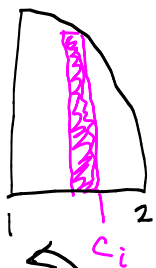
<u>Lower sum</u>: The sum of areas of approximating rectangles, with the height of each rectangle equal to the *minimum* y-value on that subinterval.

As $n \to \infty$, both the lower sum and the upper sum will approach the same limit. The area under the curve is equal to this limit.

Example 6: Using the limit definition for area, calculate the area of the region bounded by the graph of $f(x) = 4 - x^2$, the *x*-axis, and the vertical lines $x = 1$ and $x = 2$.

(See Example 6, p. 261 of Larson book…..you'll need to use the aforementioned summation formulas. Do you get the same result using the left endpoints?)





Find x-intercepts: set $y=0$: $0 = 4 - x^2$

$x^2 = 4$

$x = \pm 2$

Using right endpoints.

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

use c_i to get height of the i^{th} rectangle

$c_i = a + i\Delta x$ — we'll use $\sum_{i=1}^{n}$

Note: $c_n = a + n\Delta x$

$= a + n\left(\frac{b-a}{n}\right) = a + b - a = b$

$c_i = a + i\Delta x$

$= 1 + i\left(\frac{1}{n}\right) = 1 + \frac{i}{n}$

height: $f(c_i) = f\left(1 + \frac{i}{n}\right)$

$= 4 - \left(1 + \frac{i}{n}\right)^2$

$= 4 - \left(1 + \frac{2i}{n} + \frac{i^2}{n^2}\right)$

$= 3 - \frac{2i}{n} - \frac{i^2}{n^2}$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(3 - \frac{2i}{n} - \frac{i^2}{n^2}\right)\left(\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\frac{3}{n} - \frac{2i}{n^2} - \frac{i^2}{n^3}\right)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{3}{n} - \sum_{i=1}^{n}\frac{2i}{n^2} - \sum_{i=1}^{n}\frac{i^2}{n^3}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}3 - \frac{2}{n^2}\sum_{i=1}^{n}i - \frac{1}{n^3}\sum_{i=1}^{n}i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}(3n) - \frac{2}{n^2}\cdot\frac{n(n+1)}{2} - \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[3 - \frac{n+1}{n} - \frac{2n^2+3n+1}{6n^2}\right]$$

$$= \lim_{n\to\infty}\left[3 - \frac{n}{n} - \frac{1}{n} - \frac{2n^2}{6n^2} - \frac{3n}{6n^2} - \frac{1}{6n^2}\right]$$

$$= \lim_{n\to\infty}\left[3 - 1 - \frac{1}{n} - \frac{1}{3} - \frac{1}{2n} - \frac{1}{6n^2}\right]$$

$$= 2 - 0 - \frac{1}{3} - 0 - 0$$

$$= 2 - \frac{1}{3} = \frac{6}{3} - \frac{1}{3} = \boxed{\frac{5}{3}}$$