1.2: Finding Limits Graphically and Numerically

Limit of a function:

Definition of a Limit:

$$\lim_{x \to a} f(x) = L$$

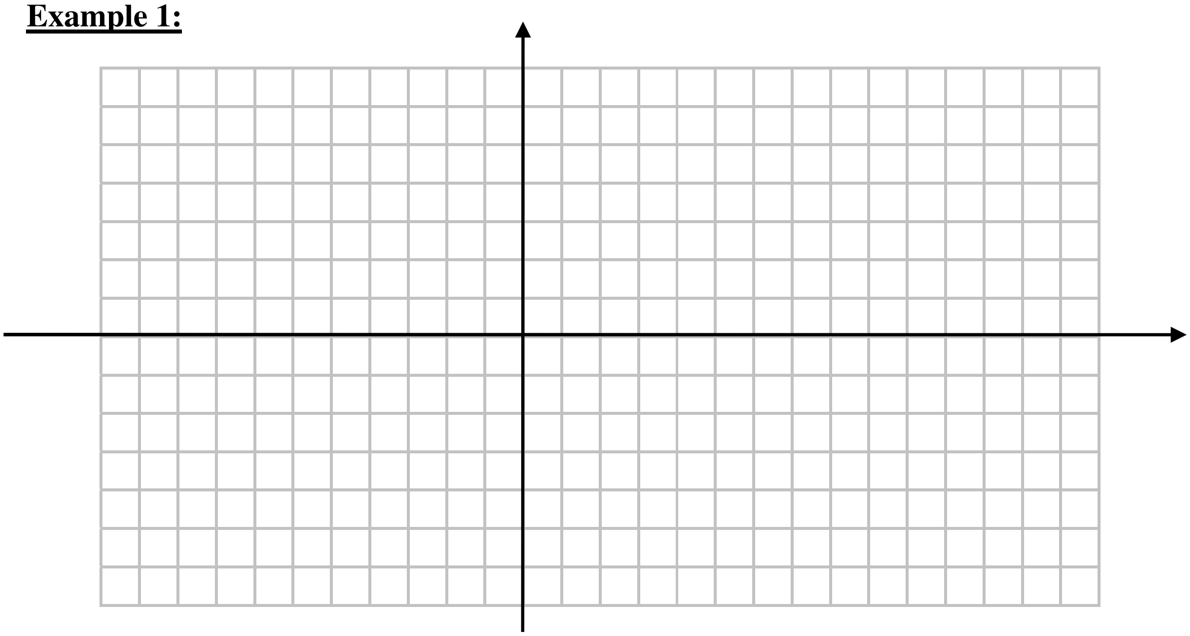
The statement above means that we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but not equal to a.

We read this as "the limit of $f(x)$, as x approaches a, is equal to L."

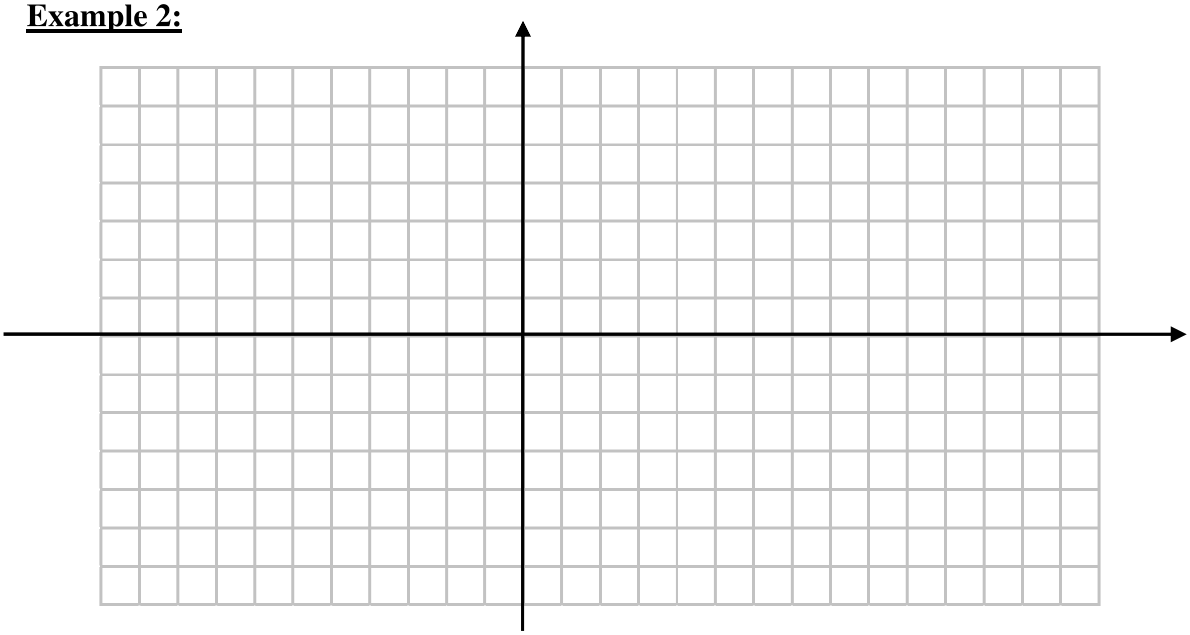
Alternative notation: $f(x) \to L$ as $x \to a$. ($f(x)$ approaches L as x approaches a)

Finding limits from a graph:

Example 1:

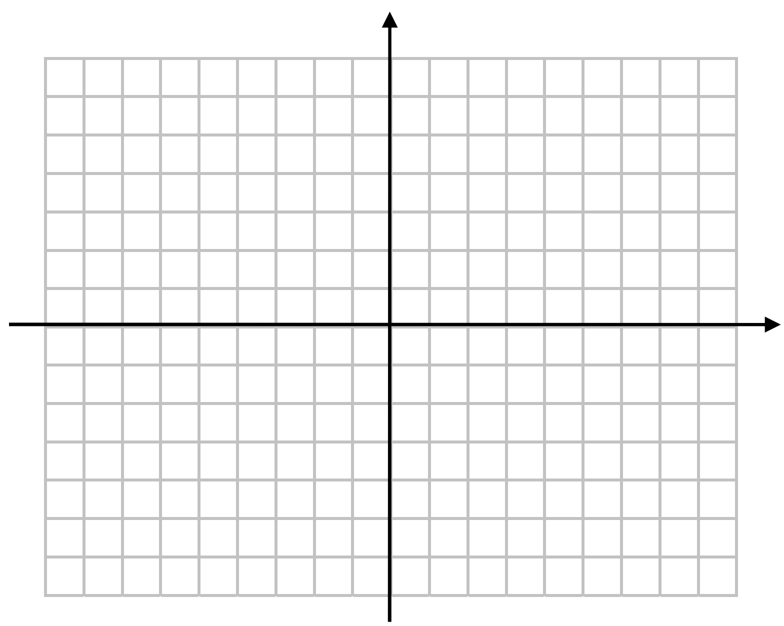


Example 2:



Example 3: Graph the function. Use the graph to determine $\lim_{x\to 2} f(x)$ and $\lim_{x\to -1} f(x)$.

$$f(x)=\begin{cases} x-1 & \text{if } x\geq 2 \\ x^2 & \text{if } x<2 \end{cases}$$



Finding limits numerically:

<u>Example 4:</u> For the function $f(x) = \dfrac{x-5}{x^2-25}$, make a table of function values corresponding to values of x near 5.

Use the table to estimate the value of $\lim_{x \to 5} \dfrac{x-5}{x^2-25}$.

<u>Example 5:</u> Make a table of values and use it to estimate $\lim_{x \to 0} \sin\left(\dfrac{1}{x}\right)$.

<u>Common reasons $\lim_{x \to c} f(x)$ may not exist:</u>

1. $f(x)$ approaches a different value when approached from the left of c, compared to when approached from the right of c.

2. $f(x)$ increases or decreases without bound as x approaches c.

3. $f(x)$ oscillates between two values as x approaches c.

The formal (epsilon-delta) definition of a limit: (Section 1.2 p. 4)

<u>Definition</u>:

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

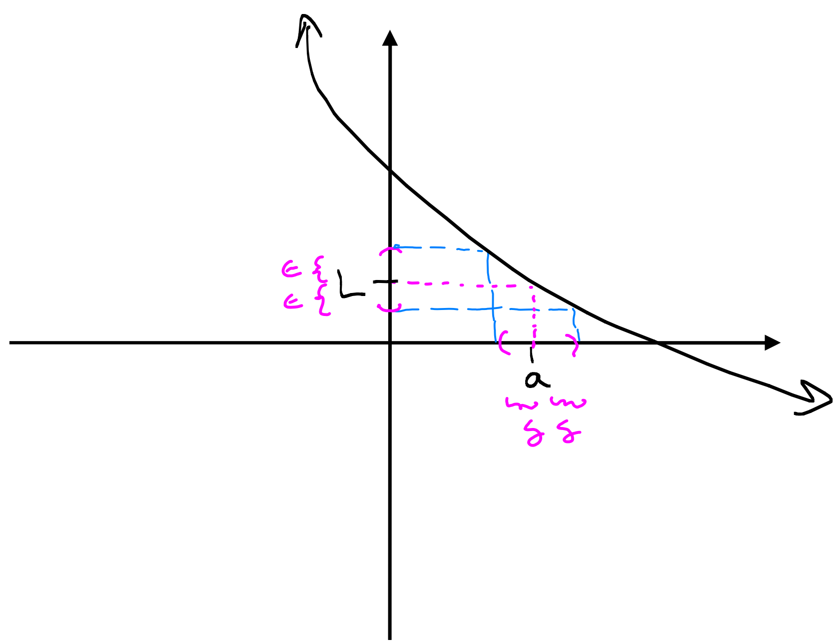
$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

$\in$ or ε : epsilon
δ : delta (lower-case)

<u>Example 6:</u>

Here,

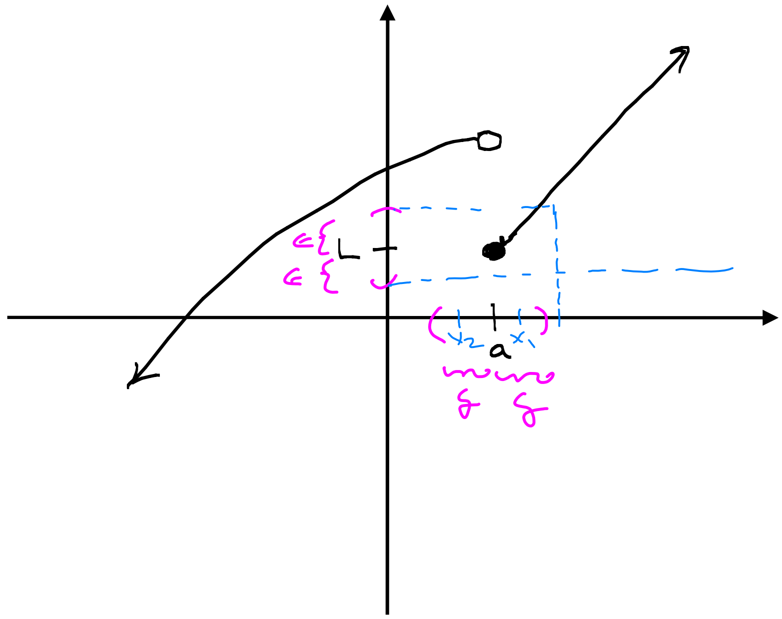
$$\lim_{x \to a} f(x) = L$$



Example 7:

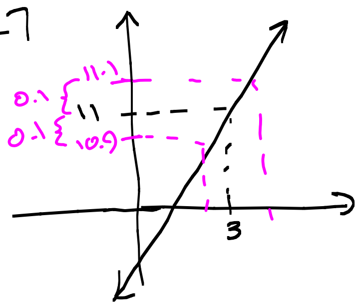
$$\lim_{x \to a} f(x) \neq L$$

$f(x_2)$ is not within ϵ of L.



Example 8: How close to 3 must we take x so that $6x-7$ is within 0.1 of 11?

$y = 6x-7$



$x = 3 \Rightarrow 6(3)-7 = 18-7 = 11$

$6x-7 = 11.1$

$6x = 18.1$

$x = \frac{18.1}{6} \approx 3.01\overline{6}$

$6x-7 = 10.9$

$6x = 17.9$

$x = \frac{17.9}{6} = 2.98\overline{3}$

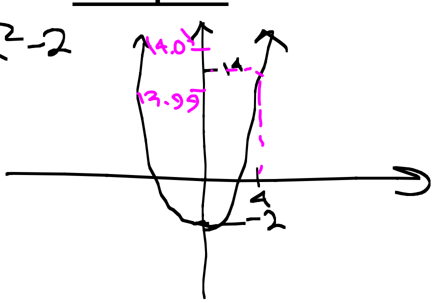
$3 - 2.98\overline{3} = 0.01\overline{6}$

Take δ = anything $< 0.01\overline{6}$

$\boxed{x \text{ must be within } 0.01\overline{6} \text{ of } 3}$

Example 9: How close to 4 must we take x so that x^2-2 is within 0.01 of 14?

$y = x^2-2$



$x = 4 \Rightarrow x^2-2 = 4^2-2 = 14$

Suppose $x^2-2 = 14.01$

$x^2 = 16.01$

$x = \pm\sqrt{16.01}$

$x \approx 4.0012498$

Suppose $x^2-2 = 13.99$

$x^2 = 15.99$

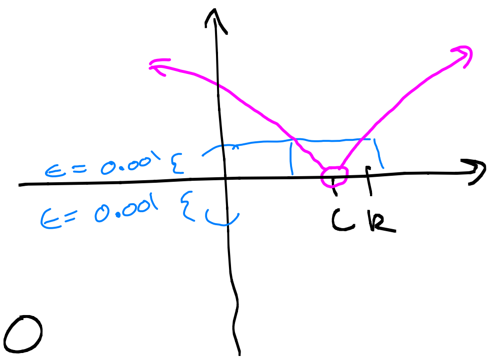
$x = \sqrt{15.99} \approx 3.9987498$

$4 - x \approx 0.001250195$

$\boxed{\text{Choose } x \text{ within } 0.0012498 \text{ of } 4}$

1.2 #68 True or False:

If $\lim_{x \to c} f(x) = 0$, then there exists k such that $f(k) < 0.001$



for every $\epsilon > 0$, there must be a $\delta > 0$ such that $0 < |x - c| < \delta$ implies that $|f(x) - L| < \epsilon$.

$L = 0$

want $|f(x) - L| < \epsilon$

$|f(x) - 0| < \epsilon$

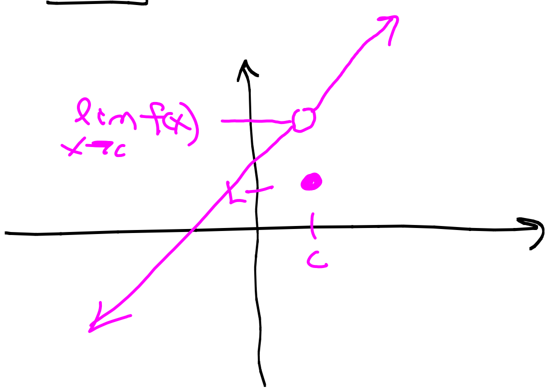
$|f(x)| < \epsilon$

Consider $\epsilon = 0.001$. Because $\lim_{x \to c} f(x) = 0$, there must be a $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - 0| < 0.001$.

So, choose any k such that $0 < |k - c| < \delta$. Then automatically $|f(k)| < 0.001$.

True or False:

#69 If $f(c) = L$, then $\lim_{x \to c} f(x) = L$.



Suppose $f(c) = L$.

Then $\lim_{x \to c} f(x) \neq L$

$\boxed{\text{False}}$

Fact about absolute values:

$$|AB| = |A||B|$$

Example 10: Prove that $\lim_{x\to 4}(2x-5)=3$ using the definition of a limit.

Scratchwork Let $\epsilon > 0$

We want

$|f(x)-L| = |\underbrace{(2x-5)-3}_{f(x)-L}| < \epsilon$

$|2x-8| < \epsilon$

$|2(x-4)| < \epsilon$

$|2||x-4| < \epsilon$

$2|x-4| < \epsilon$

$|x-4| < \frac{\epsilon}{2}$

So, let $\delta = \frac{\epsilon}{2}$

Proof: Let $\epsilon > 0$. Then choose $\delta = \frac{\epsilon}{2}$.

Suppose $0 < |x-4| < \delta$.

(We now need to show that $|f(x)-L| < \epsilon$.)

$|(2x-5)-3| = |2x-8| = |2(x-4)|$

$= |2||x-4| = 2|x-4| < 2\delta = 2\left(\frac{\epsilon}{2}\right) = \epsilon.$ $\blacksquare$

Example 11: Prove that $\lim_{x\to -3}(5x+1)=-14$ using the definition of a limit.

Scratchwork:

We want $|f(x)-L| < \epsilon$

$|(5x+1)-(-14)| < \epsilon$

$|5x+1+14| < \epsilon$

$|5x+15| < \epsilon$

$|5(x+3)| < \epsilon$

$5|x+3| < \epsilon$

$|x+3| < \frac{\epsilon}{5}$

So choose $\delta = \frac{\epsilon}{5}$

Proof: Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{5}$.

Suppose $0 < |x+3| < \delta$.

We must show that $|(5x+1)-(-14)| < \epsilon$.

$|(5x+1)-(-14)| = |5x+1+14| = |5x+15|$

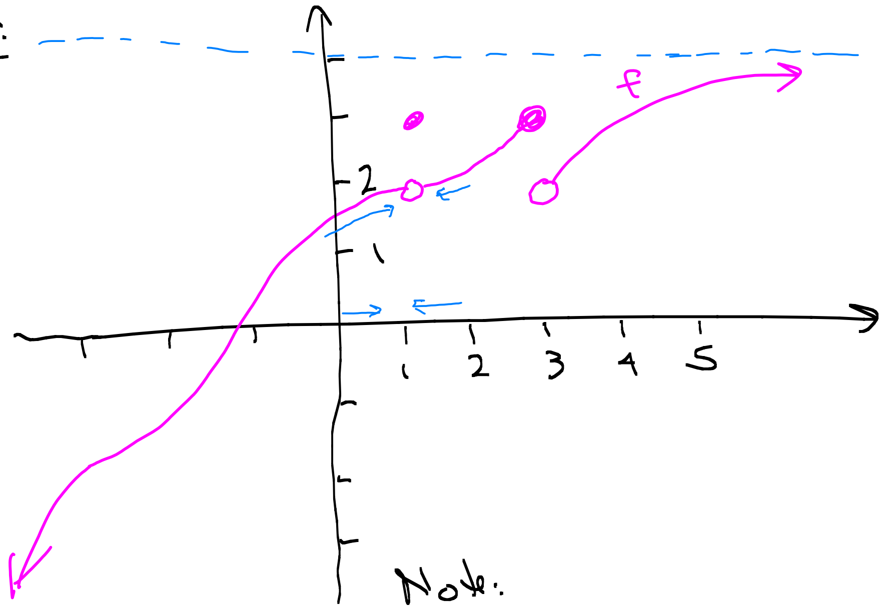
$= |5(x+3)| = |5||x+3| = 5|x+3|$

$< 5\delta = 5\left(\frac{\epsilon}{5}\right) = \epsilon.$ $\blacksquare$

Example 12: Prove that $\lim_{x \to 3} x^2 = 9$ using the definition of a limit.

Example 13: Prove that $\lim_{x \to 2}(x^2 - x + 6) = 8$ using the definition of a limit.

Ex:



$\lim_{x \to 1} f(x) \approx \boxed{1.9}$

$\lim_{x \to 2} f(x) = 2$

$\lim_{x \to 3} f(x)$ Does not exist

$\lim_{x \to \infty} f(x) = \boxed{4}$

Note:

$f(1) = 3$

$f(3) = 3$

Ex: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

x	$f(x) = \frac{x^2-4}{x-2}$
1.5	3.5
1.8	3.8
1.9	3.9
1.99	3.99
1.999	3.999
2.5	4.5
2.3	4.3
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

From table,

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

How to calculate without table?

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x \to 2} (x+2) = 2 + 2$$

$$= \boxed{4}$$

Note: $\frac{0}{0} \neq 1$

$\frac{0}{0}$ is undefined. In regards to limits, the $\frac{0}{0}$ pattern is called an indeterminate form.

Ex: Draw a graph of a function f where

$f(2)$ is undefined and $\lim_{x \to 2} = 5$,

also $f(-2) = 1$

and $\lim_{x \to -2} f(x)$

does not exist.