## 1.5: Infinite Limits

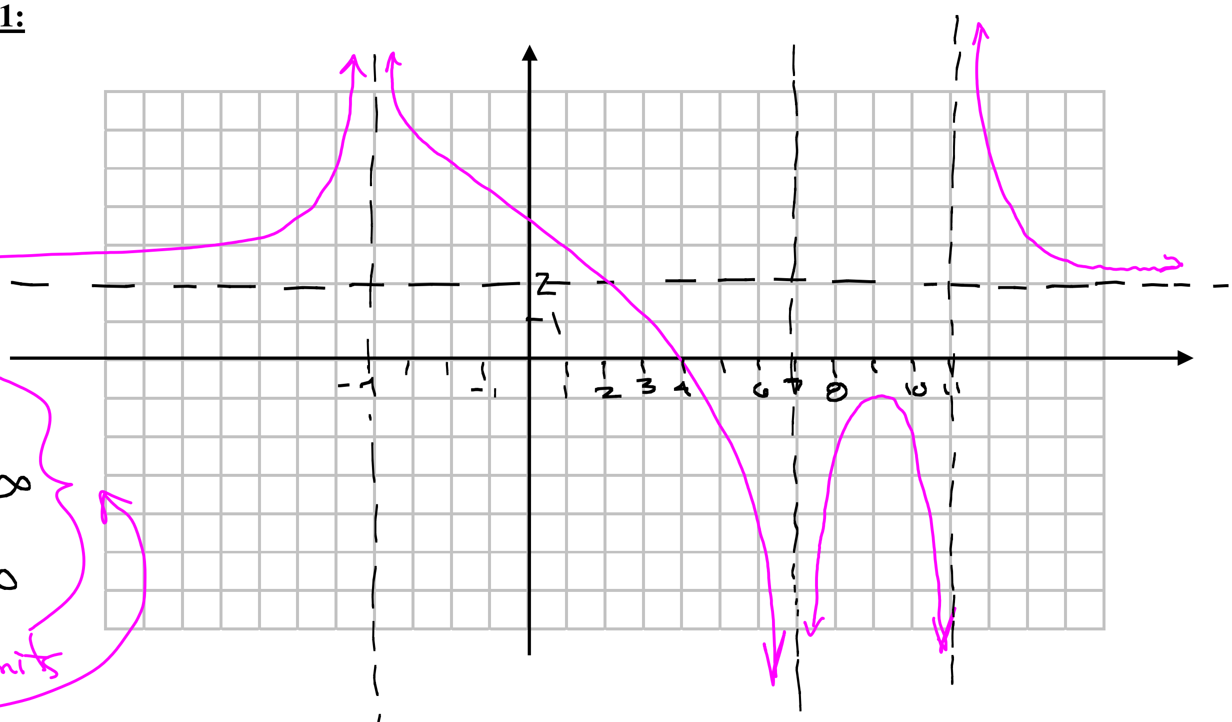
There are two types of limits involving infinity. $\lim_{x\to\infty} f(x)$ $\lim_{x\to-\infty} f(x)$

Limits at infinity, written in the form $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$, are related to horizontal asymptotes and will be covered in Section 3.5, as we learn to graph functions.

Infinite limits take the form of statements like $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. Infinite limits can result in vertical asymptotes, also important in graphing functions.

**Determining infinite limits from a graph:**

**Example 1:**




$\lim_{x\to -4^-} f(x) = +\infty$

$\lim_{x\to -4^+} f(x) = +\infty$

$\lim_{x\to -4} f(x) = +\infty$

these limits do not exist

$\lim_{x\to 7^-} f(x) = -\infty$

$\lim_{x\to 7^+} f(x) = -\infty$

$\lim_{x\to 7} f(x) = -\infty$

$\lim_{x\to 11^-} f(x) = -\infty$

$\lim_{x\to 11^+} f(x) = +\infty$ (or just $\infty$)

$\lim_{x\to 11} f(x)$ does not exist

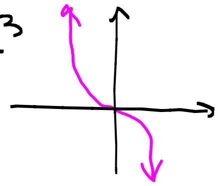
none of the above limits exist.

$\lim_{x\to\infty} f(x) = 2$

$\lim_{x\to-\infty} f(x) = 2$

these 2 limits exist.

Ex: $f(x) = -x^3$



$\lim_{x \to \infty} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = +\infty$

these limits do not exist

**Determining infinite limits from a table of values:**

**<u>Example 2:</u>** Use a table of values to determine $\lim_{x \to 2} f(x) = \dfrac{x+6}{x-2}$.

| $x$ | $f(x) = \dfrac{x+6}{x-2}$ |
|---|---|
| 1.9 | |
| 1.95 | |
| 1.99 | |
| 1.999 | |
| 1.9999 | |
| ⋮ | $\downarrow -\infty$ |

| $x$ | $f(x) = \dfrac{x+6}{x-2}$ |
|---|---|
| 2.10 | |
| 2.05 | |
| 2.01 | |
| 2.001 | |
| 2.0001 | $\downarrow +\infty$ |
| ⋮ | |

$\boxed{\text{So, } \lim_{x \to 2} f(x) \text{ does not exist}}$

**<u>Example 3:</u>** Use a table of values to determine $\lim_{x \to 2} f(x) = \dfrac{x+6}{(x-2)^2}$.

| $x$ | $f(x)$ |
|---|---|
| 1.9 | |
| 1.95 | |
| 1.99 | |
| 1.999 | |
| ⋮ | $\downarrow +\infty$ |

| $x$ | $f(x)$ |
|---|---|
| 2.10 | |
| 2.05 | |
| 2.01 | |
| 2.001 | $\downarrow +\infty$ |
| ⋮ | |

$\lim_{x \to 2} \dfrac{x+6}{(x-2)^2} = +\infty$ (does not exist)

<u>Important:</u>

Statements such as $\lim_{x \to a} f(x) = \infty$, $\lim_{x \to a} f(x) = -\infty$, or $\lim_{x \to a^+} f(x) = -\infty$ do NOT mean the limit exists. Rather, these statements mean that the limit DOES NOT EXIST. and they describe the reason that the limit fails to exist (by describing the behavior of the function near the given *x*-value).

**Formal definition of an infinite limit:**

Let *f* be a function defined on some open interval that contains the number *a*, except possibly at *a* itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive real number *M*, there exists a number $\delta > 0$ such that

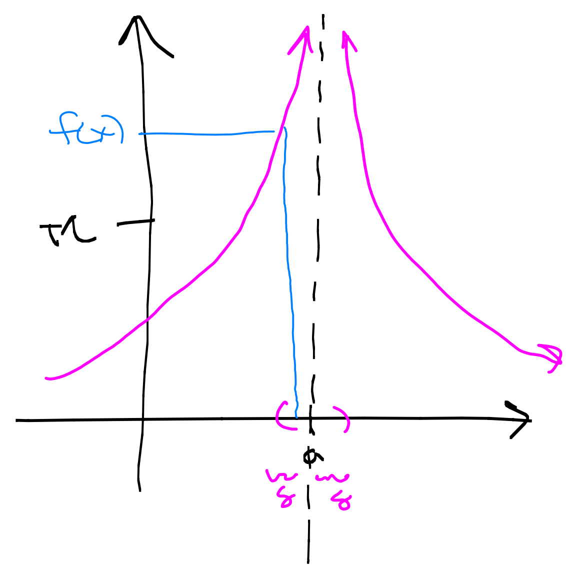
$$f(x) > M \text{ whenever } 0 < |x - a| < \delta .$$

Similarly, $\lim_{x \to a} f(x) = \cancel{\infty}\, -\infty$ means that for every negative real number *N*, there exists a number $\delta > 0$ such that

$$f(x) < N \text{ whenever } 0 < |x - a| < \delta .$$

**Example 4:** Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Picture for defn, not for Ex 4



**Evaluating infinite limits from an equation:**

**Example 5:** Determine $\lim_{x\to 4} \frac{x-8}{x-4}$.

$\lim_{x\to 4} \frac{x-8}{x-4}$

Direct Substitution

$\frac{4-8}{4-4} \Rightarrow \frac{-4}{0}$

limit does not exist

try $x=3.99 \Rightarrow \frac{3.99-8}{3.99-4} \Rightarrow \frac{-4}{-\text{tiny}} \Rightarrow +\text{huge}$

So, $\boxed{\lim_{x\to 4^-} \frac{x-8}{x-4} = +\infty}$

$\lim_{x\to 4^+} \frac{x-8}{x-4}$

try $x=4.01 \Rightarrow \frac{4.01-8}{4.01-4} \Rightarrow \frac{-4}{+\text{tiny}} \Rightarrow -\text{huge}$

So, $\boxed{\lim_{x\to 4^+} \frac{x-8}{x-4} = -\infty}$

$\therefore$ $\boxed{\lim_{x\to 4} \frac{x-8}{x-4} \text{ does not exist}}$

**Example 6:** Determine $\lim_{x\to 2^-} \frac{x-8}{x^2-4}$.

$\boxed{\lim_{x\to 2^-} \frac{x-8}{x^2-4} = +\infty}$ (does not exist)

$x=1.99 \Rightarrow \frac{1.99-8}{(1.99)^2-4} \Rightarrow \frac{-6}{-\text{tiny}} \Rightarrow +\text{huge}$

**Example 7:** Determine $\lim_{x\to 3} \frac{x^3-2}{(x-3)^2}$.

From left

$x=2.99 \Rightarrow \frac{(2.99)^3-2}{(2.99-3)^2} \Rightarrow \frac{27-2}{(-\text{tiny})^2} \Rightarrow \frac{25}{+\text{tiny}} \Rightarrow +\text{huge}$

From right:

$x=3.01$ $\frac{(3.01)^3-2}{(3.01-3)^2} \Rightarrow \frac{27-2}{(+\text{tiny})^2} \Rightarrow \frac{25}{+\text{tiny}} \Rightarrow +\text{huge}$

So, $\lim_{x\to 3} \frac{x^3-2}{(x-3)^2} = \boxed{+\infty}$

(does not exist)

**Example 8:** Determine $\lim_{x\to \pi^+} \frac{\sin\left(\frac{x}{3}\right)}{1+\cos x}$.

$\lim_{x\to \pi^+} \frac{\sin(x/3)}{1+\cos x} = \boxed{+\infty \quad \text{(does not exist)}}$

$x=3.16 \Rightarrow \frac{\sin \pi/3}{1+\cos(3.16)} \Rightarrow \frac{\sqrt{3}/2}{1-0.99} \Rightarrow \frac{\sqrt{3}/2}{+\text{tiny}} \Rightarrow +\text{huge}$

Note: $\lim_{x\to \pi^-} \frac{\sin(x/3)}{1+\cos x} = +\infty$ also

**Example 9:** Determine $\lim_{x\to \pi/6} \tan(3x)$.

$\lim_{x\to \pi/6} \tan(3x) = \lim_{x\to \pi/6} \frac{\sin(3x)}{\cos(3x)}$ $\boxed{\text{does not exist}}$

Direct Substitution:

$\frac{\sin(3\pi/6)}{\cos(3\pi/6)} \Rightarrow \frac{\sin(\pi/2)}{\cos(\pi/2)} \Rightarrow \frac{1}{0}$

As $x \to \frac{\pi}{6}^-$,

$y \to \frac{\sin(3\pi/6)}{+\text{tiny}} \Rightarrow \frac{1}{+\text{tiny}} \Rightarrow +\text{huge}$

As $x \to \frac{\pi}{6}^+$,

$y \to \frac{1}{-\text{tiny}} \Rightarrow -\text{huge}$

**Vertical asymptotes:**

<u>Vertical Asymptotes</u>:

The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

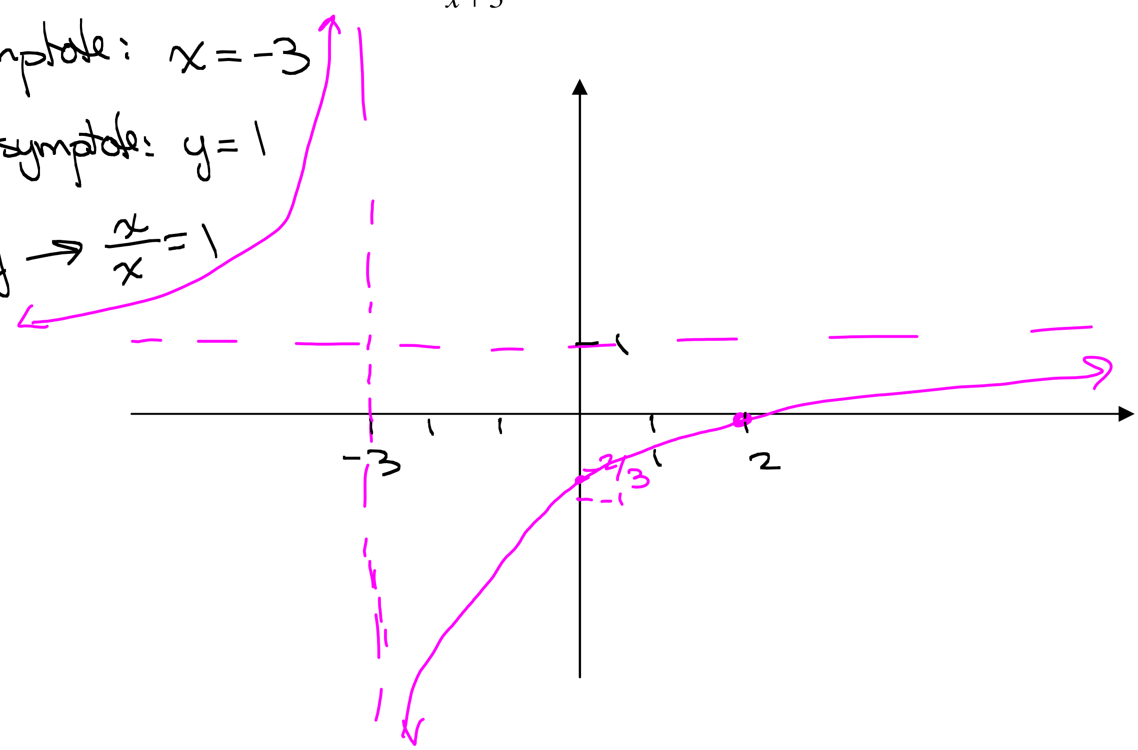
$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

**<u>Example 10:</u>** Determine the asymptotes of $f(x) = \dfrac{x-2}{x+3}$. Sketch the graph.

Vertical asymptote: $x = -3$

Horizontal asymptote: $y = 1$

as $x \to \pm\infty$, $y \to \frac{x}{x} = 1$



right of 1

$x = 1.01 \Rightarrow$

$y = \frac{3}{(1.01)^2 - 1}$

$\Rightarrow \frac{3}{+\text{tiny}}$

$\Rightarrow +\infty$

**Example 11:** Determine the vertical asymptotes of $f(x) = \frac{3}{x^2 - 1}$. Sketch the graph.

$f(x) = \frac{3}{(x+1)(x-1)}$

Vertical asymptotes:

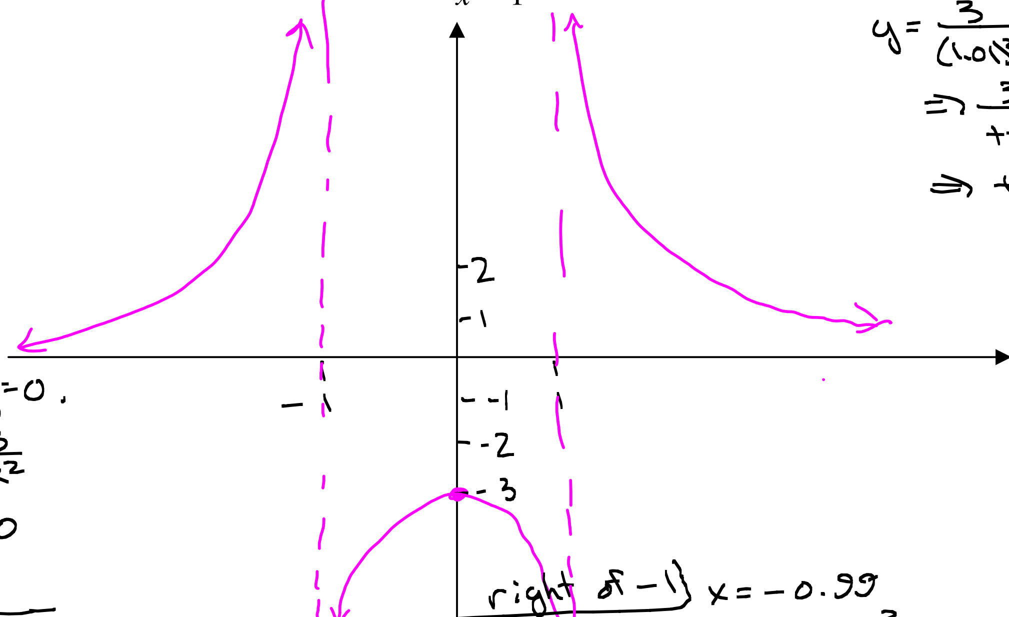
$x = -1,\ x = 1$

Horizontal asymptote: $y = 0$

(From algebra class, if deg(denom) > deg(num), horizontal asymptote is $y = 0$.

Or: as $x \to \pm\infty$, $y \to \frac{3}{x^2}$

$\Rightarrow \frac{3}{(\pm \text{huge})^2} \to 0$



left of $-1$ ) $x = -1.01 \Rightarrow y = \frac{3}{(-1.01)^2 - 1}$

$\Rightarrow \frac{3}{+\text{tiny}} \Rightarrow +\text{huge} \Rightarrow \lim_{x \to -1^-} \frac{3}{x^2 - 1} = +\infty$

right of $-1$ ) $x = -0.99$

$f(-0.99) = \frac{3}{(-.99)^2 - 1} \Rightarrow \frac{3}{-\text{tiny}} \Rightarrow -\infty$

similarly,

$\lim_{x \to 1^-} f(x) = -\infty$

**Example 12:** Determine the vertical asymptotes of $f(x) = \frac{x^2 - 9}{x^2 - 5x + 6}$. Sketch the graph.

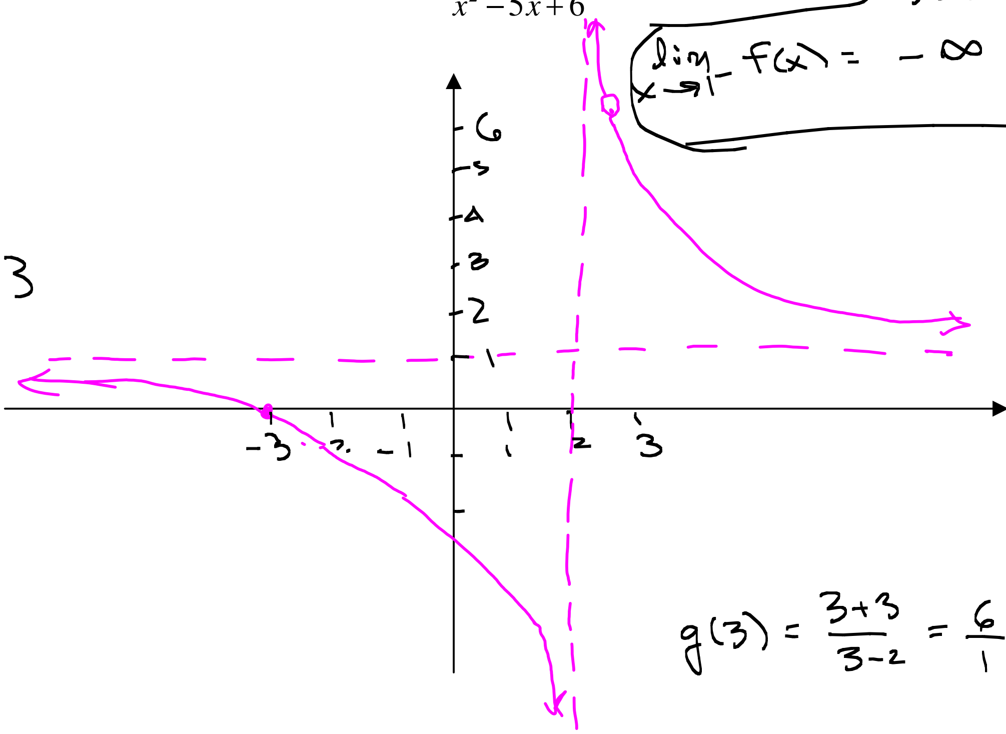
$f(x) = \frac{(x+3)(x-3)}{(x-3)(x-2)}$

x-intercept at $x = -3$

vertical asyptote $x = 2$

removable discontinuity at 3

1st graph $g(x) = \frac{x+3}{x-2}$



$\lim_{x \to 2^-} \frac{x+3}{x-2} = -\infty$

$x = 1.99 \Rightarrow \frac{1.99 + 3}{1.99 - 2} \Rightarrow \frac{5}{-\text{tiny}}$

$\Rightarrow -\text{huge}$

$g(3) = \frac{3+3}{3-2} = \frac{6}{1} = 6$

$\lim_{x \to 2^+} \frac{x+3}{x-2} = +\infty$

$x = 2.01 \Rightarrow \frac{2.01 + 3}{2.01 - 2} \Rightarrow \frac{5}{+\text{tiny}} \Rightarrow +\text{huge}$

Horizontal asymptote: $y = 1$

As $x \to \pm\infty$, $y \to \frac{x}{x} = 1$