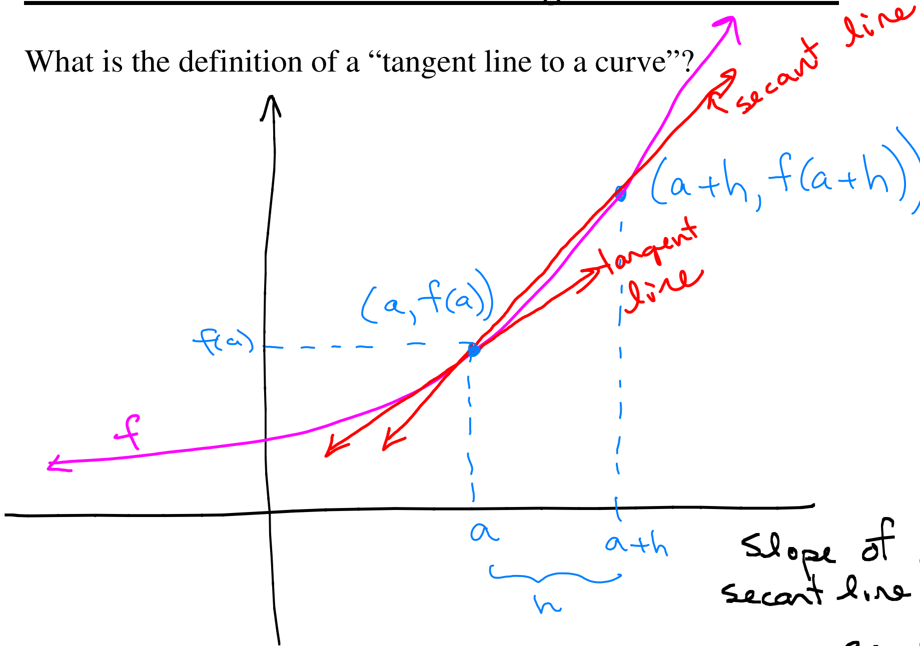
2.1: The Derivative and the Tangent Line Problem

What is the definition of a "tangent line to a curve"?

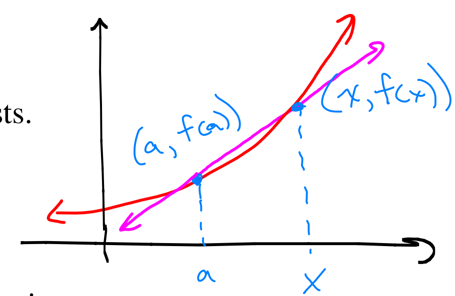


$$\text{Slope of secant line} = \frac{y_2-y_1}{x_2-x_1} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

To answer the difficulty in writing a clear definition of a tangent line, we can define it as the limiting position of the secant line as the second point approaches the first.

<u>Definition</u>: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{provided this limit exists.}$$



Equivalently,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{provided this limit exists.}$$

<u>Note</u>: If the tangent line is vertical, this limit does not exist. In the case of a vertical tangent, the equation of the tangent line is $x = a$.

<u>Note</u>: The slope of the tangent line to the graph of f at the point $(a, f(a))$ is also called the <u>slope of the graph</u> of f at $x = a$.

<u>How to get the second expression for slope</u>: Instead of using the points $(a, f(a))$ and $(x, f(x))$ on the secant line and letting $x \to a$, we can use $(a, f(a))$ and $(a+h, f(a+h))$ and let $h \to 0$.

here, $a = 3$

Example 1: Find the slope of the curve $y = 4x^2 + 1$ at the point $(3, 37)$. Find the equation of the tangent line at this point.

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{4(3+h)^2+1-[4(3)^2+1]}{h}$$

$$= \lim_{h\to 0} \frac{4(9+6h+h^2)+1-37}{h} = \lim_{h\to 0} \frac{\cancel{36}+24h+4h^2-\cancel{36}}{h}$$

$$= \lim_{h\to 0} \frac{24h+4h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(24+4h)}{\cancel{h}} = \lim_{h\to 0} (24+4h)$$

$$= 24 + 4(0) = 24 + 0 = 24$$

Find eqn of tangent line:

$y - y_1 = m(x - x_1)$

$\left.\begin{matrix}(x_1,y_1)=(3,37)\\ m=24\end{matrix}\right\} \Rightarrow y - 37 = 24(x-3)$

$y - 37 = 24x - 72$

$\boxed{y = 24x - 35}$

Example 2: Find an equation of the tangent line to the curve $y = x^3$ at the point $(1,1)$.

Alternative definition:

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^3-a^3}{x-a}$$

$$= \lim_{x\to a} \frac{\cancel{(x-a)}(x^2+ax+a^2)}{\cancel{x-a}} = \lim_{x\to a} (x^2+ax+a^2)$$

Recall: Sum and Difference of 2 cubes

$x^3 - a^3 = (x-a)(x^2+ax+a^2)$

$x^3 + a^3 = (x+a)(x^2-ax+a^2)$

For $a = 1$,

$m = \lim_{x\to 1} (x^2 + 1x + 1^2) = \lim_{x\to 1} (x^2+x+1) = 1^2+1+1 = 3$. This is the slope

Find eqn: $y - y_1 = m(x-x_1) \Rightarrow y - 1 = 3(x-1) \Rightarrow y - 1 = 3x - 3 \Rightarrow \boxed{y = 3x - 2}$

Example 3: Determine the equation of the tangent line to $f(x) = \sqrt{x}$ at the point where $x = 2$.

$$\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} = \lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} \cdot \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}$$

$$= \lim_{x\to 2} \frac{\cancel{x-2}}{\cancel{(x-2)}(\sqrt{x}+\sqrt{2})}$$

$(a-b)(a+b) = a^2 - b^2$

$$= \lim_{x\to 2} \frac{1}{\sqrt{x}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4}$$

$y - y_1 = m(x - x_1)$ with $m = \frac{\sqrt{2}}{4}$, $x = 2$, $y = \sqrt{2}$

$y - \sqrt{2} = \frac{\sqrt{2}}{4}(x-2)$

$y - \sqrt{2} = \frac{\sqrt{2}}{4}x - \frac{2\sqrt{2}}{4}$

$y - \sqrt{2} = \frac{\sqrt{2}}{4}x - \frac{\sqrt{2}}{2}$

$y = \frac{\sqrt{2}}{4}x - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{1}\left(\frac{2}{2}\right)$

$y = \frac{\sqrt{2}}{4}x - \frac{\sqrt{2}}{2} + \frac{2\sqrt{2}}{2}$

$\boxed{y = \frac{\sqrt{2}}{4}x + \frac{\sqrt{2}}{2}}$

Ex 3: Using $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sqrt{2+h} - \sqrt{2}}{h} \cdot \frac{\sqrt{2+h} + \sqrt{2}}{\sqrt{2+h} + \sqrt{2}} \right]$$

$$= \lim_{h \to 0} \left[\frac{\cancel{2} + h - \cancel{2}}{h(\sqrt{2+h} + \sqrt{2})} \right] = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{2+h} + \sqrt{2})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} = \frac{1}{\sqrt{2+0} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

The derivative:

The derivative of a function f at x is the slope of the tangent line at the point $(x, f(x))$. It is also the instantaneous rate of change of the function at x.

"f prime"

Definition: The *derivative* of a function f at x is the function f' whose value at x is given by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\text{, provided this limit exists.}$$

The process of finding derivatives is called differentiation. To differentiate a function means to find its derivative.

Equivalent ways of defining the derivative:

should get a function that has x in it

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$ (Our book uses this one. It is identical to the definition above, except uses Δx in place of h.)

$$f'(x) = \lim_{w\to x}\frac{f(w)-f(x)}{w-x}$$

should get a number

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$ (Gives the derivative at the specific point where $x = a$.)

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ (Gives the derivative at the specific point where $x = a$.)

Example 4: Suppose that $g(x) = \dfrac{x^2-6x}{3}$. Determine $g'(x)$ and $g'(3)$.

Note: $g(x) = \dfrac{x^2}{3} - \dfrac{6x}{3} = \frac{1}{3}x^2 - 2x$

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0}\frac{\overbrace{\left[\tfrac{1}{3}(x+h)^2-2(x+h)\right]}^{g(x+h)} - \overbrace{\left[\tfrac{1}{3}x^2-2x\right]}^{g(x)}}{h} = \lim_{h\to 0}\frac{\tfrac{1}{3}(x^2+2xh+h^2)-2x-2h-\tfrac{1}{3}x^2+2x}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{\tfrac{1}{3}x^2}+\tfrac{2}{3}xh+\tfrac{1}{3}h^2\cancel{-2x}-2h\cancel{-\tfrac{1}{3}x^2}\cancel{+2x}}{h}$$

$$= \lim_{h\to 0}\frac{\tfrac{2}{3}xh+\tfrac{1}{3}h^2-2h}{h} = \lim_{h\to 0}\frac{\cancel{h}(\tfrac{2}{3}x+\tfrac{1}{3}h-2)}{\cancel{h}} = \lim_{h\to 0}\left(\tfrac{2}{3}x+\tfrac{1}{3}h-2\right)$$

$$= \tfrac{2}{3}x+\tfrac{1}{3}(0)-2 = \tfrac{2}{3}x-2 \Rightarrow \boxed{g'(x) = \tfrac{2}{3}x-2}$$

see next page for g'(3)

Ex 4 cont'd)

$$g'(x) = \frac{2}{3}x - 2$$

$$g'(3) = \frac{2}{3}(3) - 2 = 2 - 2 = \boxed{0}$$

Example 5: Suppose that $f(x) = \sqrt{x^2+1}$. Find the equation of the tangent line at the point where $x = 2$.

Example 6: Determine the equation of the tangent line to $f(x) = \dfrac{x-2}{x^2+1}$ at the point $\left(-2, -\dfrac{4}{5}\right)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{x+h-2}{(x+h)^2+1} - \dfrac{x-2}{x^2+1}}{h}$$

$$= \lim_{h \to 0} \left[\frac{1}{h} \cdot \left(\frac{x+h-2}{(x+h)^2+1} - \frac{x-2}{x^2+1}\right)\right]$$

$$= \lim_{h \to 0} \left[\frac{1}{h} \cdot \left(\frac{x+h-2}{(x+h)^2+1} \cdot \frac{x^2+1}{x^2+1} - \frac{x-2}{x^2+1} \cdot \frac{(x+h)^2+1}{(x+h)^2+1}\right)\right]$$

$$= \lim_{h \to 0} \left[\frac{1}{h}\left(\frac{x^3+x+x^2h+h-2x^2-2}{((x+h)^2+1)(x^2+1)} - \frac{(x-2)(x^2+2xh+h^2+1)}{(x^2+1)((x+h)^2+1)}\right)\right]$$

$$= \lim_{h \to 0} \left[\frac{1}{h}\left(\frac{\cancel{x^3}+\cancel{x}+x^2h+h-\cancel{2x^2}-\cancel{2}-(\cancel{x^3}+2x^2h+xh^2+\cancel{x}-\cancel{2x^2}-4xh-2h^2-\cancel{2})}{((x+h)^2+1)(x^2+1)}\right)\right]$$

$$= \lim_{h \to 0} \left[\frac{1}{h}\left(\frac{x^2h + h - 2x^2h - xh^2 + 4xh + 2h^2}{((x+h)^2+1)(x^2+1)}\right]$$

See next Page

Ex 6 cont'd

$$f'(x) = \lim_{h\to 0}\left[\frac{1}{h}\cdot\frac{-x^2h + h - xh^2 + 4xh + 2h^2}{((x+h)^2+1)(x^2+1)}\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{\cancel{h}}\cdot\frac{\cancel{h}(-x^2+1-xh+4x+2h)}{((x+h)^2+1)(x^2+1)}\right]$$

$$= \lim_{h\to 0}\frac{-x^2+1-xh+4x+2h}{((x+h)^2+1)(x^2+1)}$$

$$= \frac{-x^2+1-x(0)+4x+2(0)}{((x+0)^2+1)(x^2+1)}$$

$$= \frac{-x^2+4x+1}{(x^2+1)^2}$$ This is the slope at the point $(x, f(x))$.

Slope: $m = f'(-2) = \frac{-(-2)^2+4(-2)+1}{((-2)^2+1)^2} = \frac{-4-8+1}{5^2} = \frac{-11}{25}$

Find eqn of tangent line:

$y - y_1 = m(x - x_1)$

Put in $y_1 = -\frac{4}{5}$, $x_1 = -2$, $m = -\frac{11}{25}$:

$$y - \left(-\frac{4}{5}\right) = -\frac{11}{25}(x-(-2))$$

$y + \frac{4}{5} = -\frac{11}{25}(x+2)$. Then write as $y = mx+b$.

Summary:

The slope of the secant line between two points is often called a difference quotient. The difference quotient of *f* at *a* can be written in either of the forms below.

$$\frac{f(x)-f(a)}{x-a} \qquad \frac{f(a+h)-f(a)}{h}.$$

Both of these give the slope of the secant line between two points: $(x, f(x))$ and $(a, f(a))$ or, alternatively, $(a, f(a))$ and $(a+h, f(a+h))$.
The slope of the secant line is also the average rate of change of *f* between the two points.

The derivative of *f* at *a* is:

1) the limit of the slopes of the secant lines as the second point approaches the point $(a, f(a))$.

2) the slope of the tangent line to the curve $y = f(x)$ at the point where $x = a$.

3) the (instantaneous) rate of change of f with respect to *x* at *a*.

4) $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ (limit of the difference quotient)

5) $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ (limit of the difference quotient)

Common notations for the derivative of $y = f(x)$:

$f'(x) \qquad \frac{d}{dx}\left(f(x)\right) \qquad y' \qquad D_x f(x) \qquad \frac{dy}{dx} \qquad Df(x)$

The notation $\frac{dy}{dx}$ was created by Gottfried Wilhelm Leibniz and means $\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}$. $\frac{\Delta y}{\Delta x}$

To evaluate the derivative at a particular number *a*, we write

$$f'(a) \text{ or } \left.\frac{dy}{dx}\right|_{x=a}$$

Differentiability:

Definition: A function *f* is *differentiable* at *a* if $f'(a)$ exists. It is *differentiable on an open interval* if it is differentiable at every number in the interval.

> Theorem: If *f* is differentiable at *a*, then *f* is continuous at *a*.

Note: The converse is not true—there are functions that are continuous at a number but not differentiable.

Note: Open intervals: (a,b), $(-\infty,a)$, (a,∞), $(-\infty,\infty)$.

Closed intervals: $[a,b]$, $(-\infty,a]$, $[a,\infty)$, $(-\infty,\infty)$.

To discuss differentiability on a closed interval, we need the concept of a *one-sided derivative*.

Derivative from the left: $\lim_{x\to a^-}\dfrac{f(x)-f(a)}{x-a}$

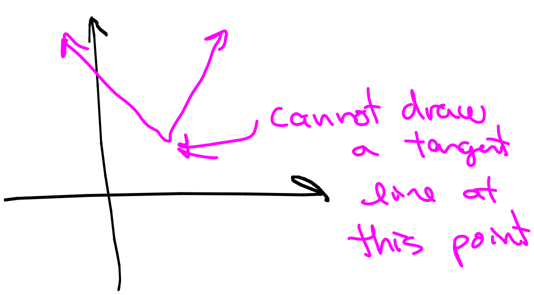
Derivative from the right: $\lim_{x\to a^+}\dfrac{f(x)-f(a)}{x-a}$

For a function *f* to be differentiable on the closed interval $[a,b]$, it must be differentiable on the open interval (a,b). In addition, the derivative from the right at *a* must exist, and the derivative from the left at *b* must exist.

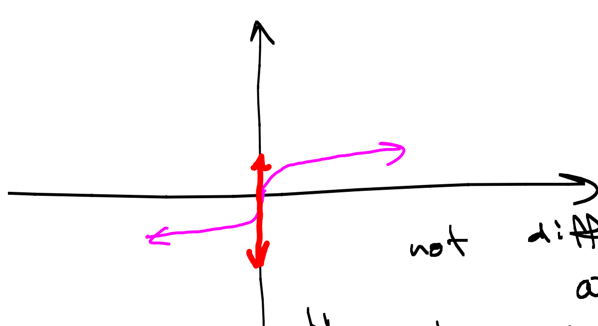
Ways in which a function can fail to be differentiable:

1. Sharp corner
2. Cusp
3. Vertical tangent
4. Discontinuity

Ex. $f(x) = \sqrt[3]{x}$



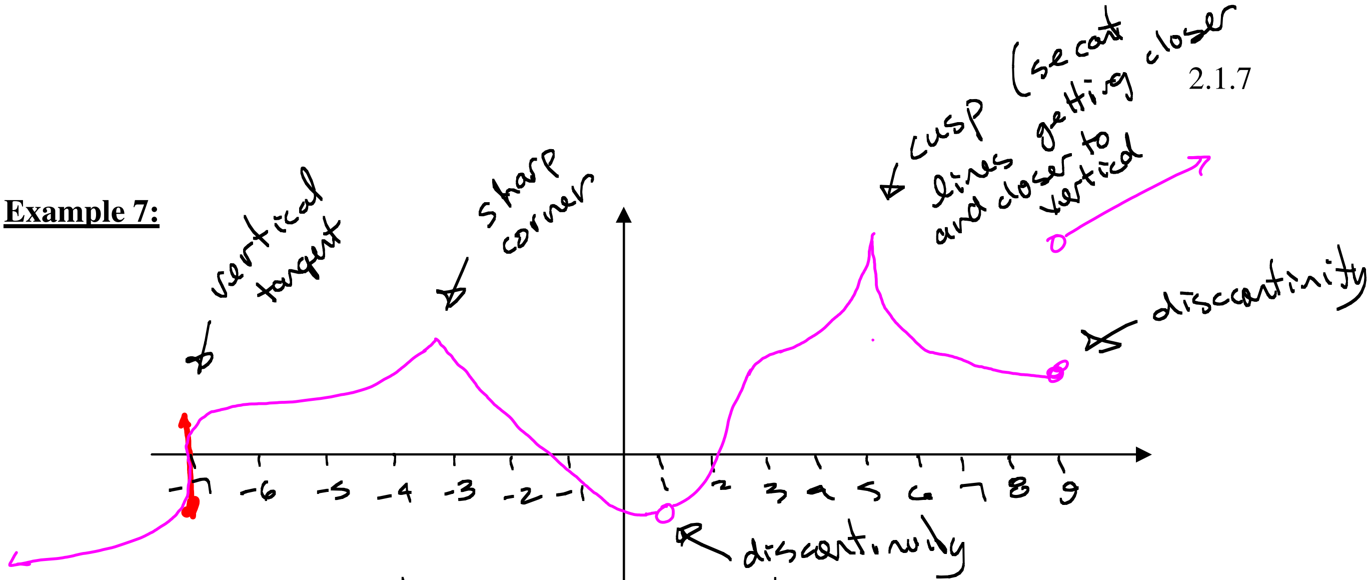
Cusp: Slopes of secant lines approach $\pm\infty$



not differentiable at 0

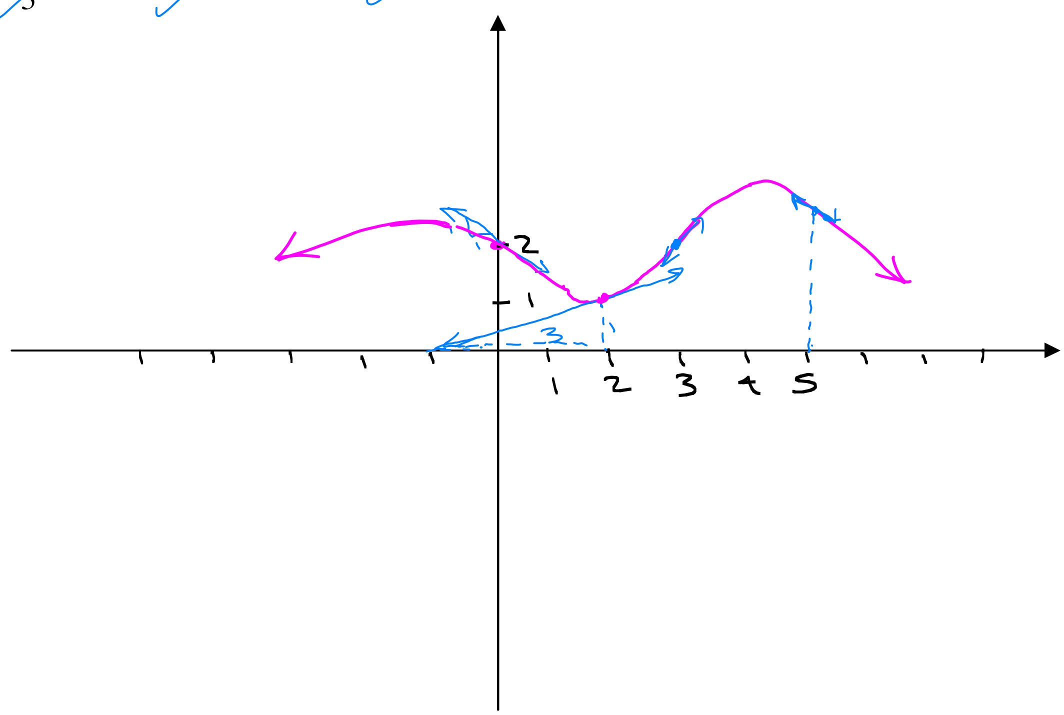
Has vertical tangent line. Slope of a vertical line is not defined.

Example 7:



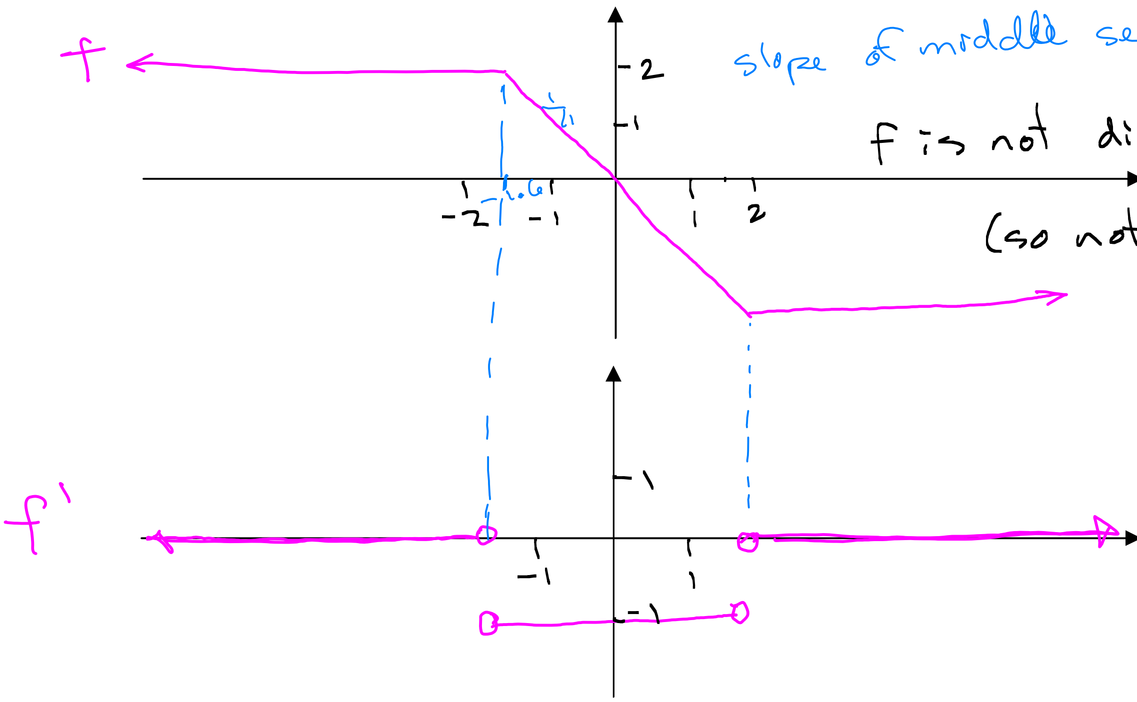
This function is not differentable at -7, -3.25, 1, 5, 9

Example 8: Sketch the graph of a function for which $f(0)=2$, $f'(0)=-1$, $f(2)=1$, $f'(2)=\frac{1}{3}$, $f'(3)>f'(2)$, and $f'(5)<0$.



Example 9: Use the graph of the function to draw the graph of the derivative.

slope of middle section = −1



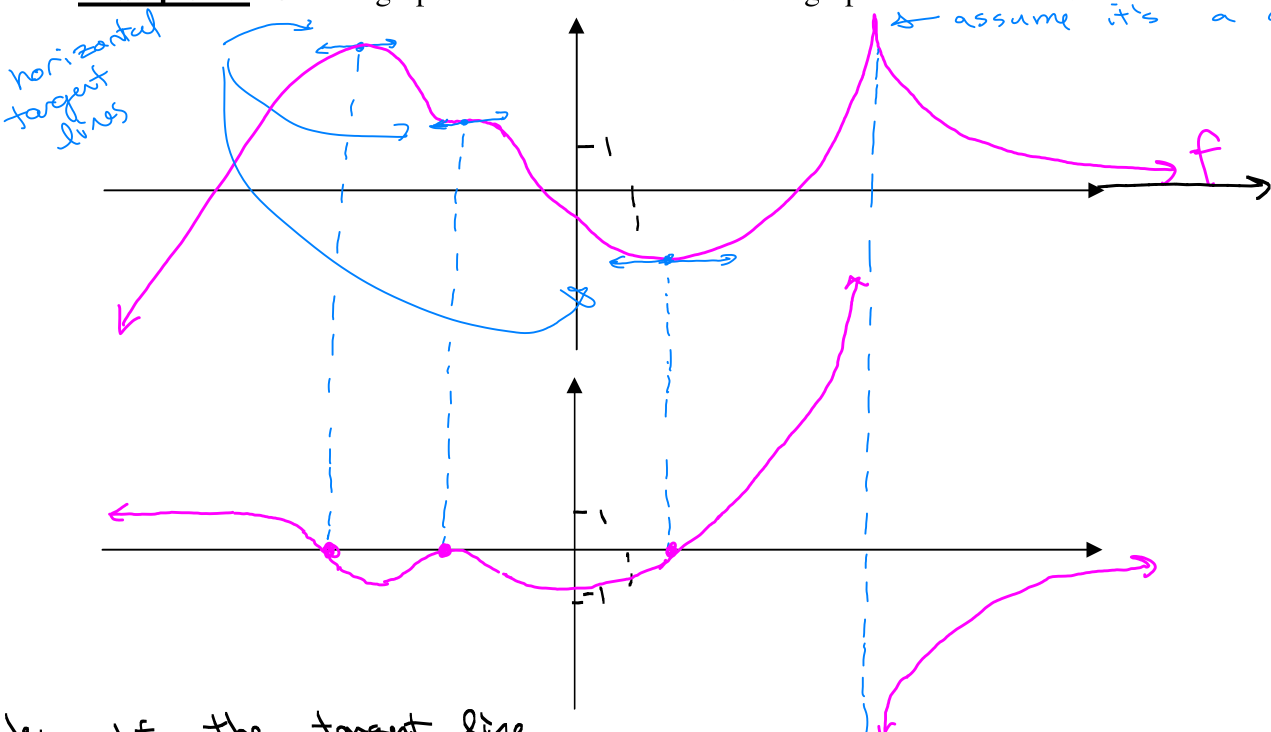
f is not differentiable at the sharp corners (so not diff'able at 2, −1.6)

so, f'(2) and f'(−1.6) are not defined.

Example 10: Use the graph of the function to draw the graph of the derivative.

horizontal tangent lines

assume it's a cusp



Note: If the tangent line is horizontal at c, then $f'(c) = 0$

Homework Qs

2.1 #19 $f(x) = x^3 - 12x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\overbrace{(x+h)^3 - 12(x+h)}^{f(x+h)} - \overbrace{(x^3 - 12x)}^{f(x)}}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)(x^2+2xh+h^2) - \cancel{12x} - 12h - x^3 + \cancel{12x}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 2x^2h + xh^2 + hx^2 + 2xh^2 + h^3 - 12h - \cancel{x^3}}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 12h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 12)}{\cancel{h}}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 12)$$

$$= 3x^2 + 3x(0) + 0^2 - 12 = \boxed{3x^2 - 12}$$

2.1 #33 Find the eqn of the line that is tangent to the graph of $f(x) = x^2$ and parallel to the line $2x - y + 1 = 0$

Find slope of given line by writing in $y = mx + b$ form.
(so find m)

Use $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$ to find $f'(x)$ in terms of x

Then set $f'(x) = m$. Solve for x.

Pt becomes $(x, f(x))$.