2.2: Basic Differentiation Rules and Rates of Change

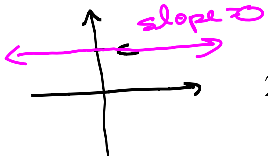
Recall:

If $y = f(x)$

$\frac{dy}{dx} = f'(x) = \frac{d}{dx}(f(x))$

Basic differentiation formulas:

$f(x) = c$



1. $\frac{d}{dx}(c) = 0$ for any constant c.

2. $\frac{d}{dx}(x^n) = nx^{n-1}$ for any real number n. (Power Rule)

3. $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$

4. $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

5. $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

Example 1: Find the derivative of $f(x) = 7$.

$f'(x) = 0$ (Rule #1)

Power Rule:

$\frac{d}{dx}(x^n) = nx^{n-1}$

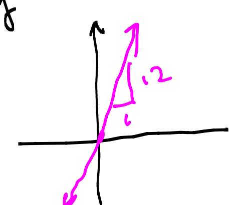
Example 2: Find the derivative of $f(x) = 5x^3 - x^7 + 12x$.

$$f'(x) = 5(3x^{3-1}) - 7x^{7-1} + 12(1x^{1-1})$$

$$= 15x^2 - 7x^6 + 12x^0$$

$$= \boxed{15x^2 - 7x^6 + 12}$$

Note: $y = 12x$



slope $= \frac{dy}{dx} = 12$

Example 3: Find the derivative of $g(x) = x^{17} + x^{3/2}$.

$$g(x) = x^{17} + x^{3/2}$$

$$g'(x) = 17x^{16} + \frac{3}{2}x^{\frac{3}{2}-1}$$

$$= 17x^{16} + \frac{3}{2}x^{\frac{3}{2}-\frac{2}{2}}$$

$$= \boxed{17x^{16} + \frac{3}{2}x^{\frac{1}{2}}}$$

Recall:
$\sqrt[n]{x} = x^{\frac{1}{n}}$
$\frac{1}{x^n} = x^{-n}$

$$\sqrt[n]{x^a} = \left(x^a\right)^{1/n} = x^{a/n}$$

Example 4: Find the derivative of $f(x) = \sqrt[5]{x} + \frac{1}{x^2}$.

$$f(x) = x^{\frac{1}{5}} + x^{-2}$$

$$f'(x) = \frac{1}{5}x^{\frac{1}{5}-1} - 2x^{-2-1} = \frac{1}{5}x^{-\frac{4}{5}} - 2x^{-3}$$

$$= \boxed{\frac{1}{5\sqrt[5]{x^4}} - \frac{2}{x^3}}$$

Example 5: Find the derivative of $f(x) = \frac{2}{\sqrt[4]{x}}$.

$$f(x) = 2x^{-1/4}$$

$$f'(x) = 2\left(-\frac{1}{4}x^{-\frac{1}{4}-1}\right) = -\frac{2}{4}x^{-5/4} = -\frac{1}{2}x^{-5/4}$$

$$= \boxed{-\frac{1}{2\sqrt[4]{x^5}}}$$

Example 6: Find the derivative of $h(x) = \left(\sqrt{x}\right)^5$.

$$h(x) = x^{5/2}$$

$$h'(x) = \frac{5}{2}x^{\frac{5}{2}-1} = \frac{5}{2}x^{3/2} = \boxed{\frac{5\sqrt{x^3}}{2}}$$

Example 7: Find the derivative of $f(x) = -\sqrt[3]{6x^4}$.

$$f(x) = -\sqrt[3]{6}\,\sqrt[3]{x^4} = -\sqrt[3]{6}\,x^{4/3}$$

$$f'(x) = -\sqrt[3]{6}\left(\frac{4}{3}x^{\frac{4}{3}-1}\right) = -\frac{4\sqrt[3]{6}}{3}x^{\frac{1}{3}} = \boxed{-\frac{4\sqrt[3]{6x}}{3}}$$

Example 8: Find the derivative of $f(x) = \frac{10}{x^4}$.

Example 9: Find the derivative of $g(x)=\dfrac{2\sqrt{x}}{7}$.

$$g(x) = \frac{2}{7}x^{1/2}$$

$$g'(x) = \frac{2}{7}\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) = \frac{2}{14}x^{-1/2} = \frac{1}{7}x^{-1/2}$$

$$= \boxed{\frac{1}{7\sqrt{x}}}$$

Example 10: Find the derivative of $f(t)=\dfrac{3}{4t^2}-\sqrt[3]{7t}$.

Example 11: Find the derivative of $f(u)=\dfrac{7u^5+u^2-9\sqrt{u}}{u^2}$.

$$f(u) = \frac{7u^5}{u^2} + \frac{u^2}{u^2} - \frac{9u^{1/2}}{u^2} = 7u^3 + 1 - 9u^{\frac{1}{2}-2} = 7u^3 + 1 - 9u^{-3/2}$$

$$f'(u) = 21u^2 + 0 - 9\left(-\frac{3}{2}u^{-\frac{3}{2}-1}\right)$$

$$= \boxed{21u^2 + \frac{27}{2}u^{-5/2}} = 21u^2 + \frac{27}{2\sqrt{u^5}}$$

Example 12: Find the equation of the tangent line to the graph of $f(x)=3x-x^2$ at the point $(-2,-10)$.

$$f'(x) = 3 - 2x$$

$$m = f'(-2) = 3 - 2(-2) = 3 + 4 = 7$$

$$y - y_1 = m(x - x_1)$$

$$y - (-10) = 7(x - (-2))$$

$$y + 10 = 7(x + 2)$$

$$y + 10 = 7x + 14$$

$$\boxed{y = 7x + 4}$$

Example 13: Find the point(s) on the graph of $f(x) = x^2 + 6x$ where the tangent line is horizontal.

Find derivative and set it equal to 0
(Slope of a horizontal line is 0)

$f'(x) = 2x + 6$

Set $f'(x) = 0$: $2x + 6 = 0$

$2x = -6$

$x = -3$

Find y-value: $f(-3) = (-3)^2 + 6(-3) = 9 - 18 = -9$

Graph is horizontal at the point $(-3, -9)$.

Definition: The *normal line* to a curve at the point *P* is defined to be the line passing through *P* that is perpendicular to the tangent line at that point.

Example 14: Determine the equation of the normal line to the curve $y = \frac{1}{x}$ at the point $\left(3, \frac{1}{3}\right)$.

Find slope of tangent line: $y = x^{-1}$

$\frac{dy}{dx} = -1x^{-1-1} = -1x^{-2} = -\frac{1}{x^2}$

$\left.\frac{dy}{dx}\right|_{x=3} = -\frac{1}{3^2} = -\frac{1}{9}$

Slope of tangent line $= -\frac{1}{9}$

Slope of normal line is $m = +\frac{9}{1} = 9$

Eqn of line: $y - y_1 = m(x - x_1)$

$y - \frac{1}{3} = 9(x - 3)$

$y - \frac{1}{3} = 9x - 27$

$y = 9x - \frac{81}{3} + \frac{1}{3}$

$\boxed{y = 9x - \frac{80}{3}}$

Derivatives of trigonometric functions:

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$

Note: Derivatives of the cofunctions have a minus sign.

Example 15: Find the derivative of $y = 2\cos x - 4\tan x$.

$$\frac{dy}{dx} = \boxed{-2\sin x - 4\sec^2 x}$$

Example 16: Find the derivative of $y = \frac{\sin x}{4} + 3x^4 + \pi^2$.

$$y = \frac{1}{4}\sin x + 3x^4 + \pi^2$$

$$\frac{dy}{dx} = \frac{1}{4}\cos x + 12x^3 + 0$$

$$= \boxed{\frac{1}{4}\cos x + 12x^3}$$

Section 2.2 cont'd

Example 17: Determine the equation of the tangent line to the graph of $y = \sec x$ at the point where $x = \frac{\pi}{4}$.

Find y-value:

$$y\Big|_{x=\frac{\pi}{4}} = \sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos(\pi/4)} = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$= \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Note: $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow \sec\left(\frac{\pi}{4}\right) = \sqrt{2}$

$$\frac{dy}{dx} = \sec x \tan x$$

$$m: \frac{dy}{dx}\Big|_{x=\frac{\pi}{4}} = \sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = (\sqrt{2})(1) = \sqrt{2}$$

Find eqn:

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)}$$

Example 18: Find the points on the curve $y = \tan x - 2x$ where the tangent line is horizontal.

Set $\frac{dy}{dx} = 0$: (slope of a horizontal line is 0).

$$\frac{dy}{dx} = \sec^2 x - 2$$

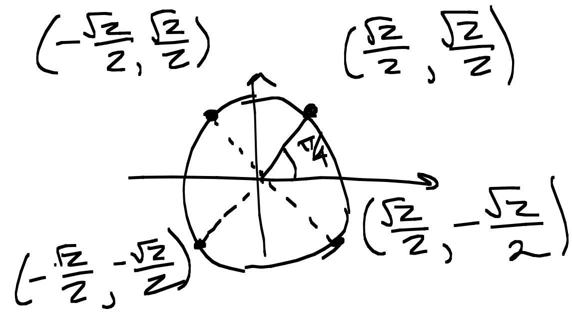
$$\sec^2 x - 2 = 0$$

$$(\sec(x))^2 = 2$$

$$\sec(x) = \pm\sqrt{2}$$

$$\cos(x) = \pm\frac{1}{\sqrt{2}}$$

$$\cos(x) = \pm\frac{\sqrt{2}}{2}$$



On $[0, 2\pi]$, $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

on $(-\infty, \infty)$

$$\boxed{x = \frac{(2k+1)\pi}{4}, \text{ where } k \text{ is any integer}}$$

(odd multiples of $\frac{\pi}{4}$)

The derivative as a rate of change:

The <u>average rate of change</u> of $y = f(x)$ with respect to x over the interval $[x_0, x_1]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}, \text{ where } h = x_1 - x_0 \neq 0.$$

This is the same as the slope of the secant line joining points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$.

The <u>instantaneous rate of change</u> (or, equivalently, just the <u>rate of change</u>) of f when $x = a$ is the slope of the tangent line to graph of f at the point $(a, f(a))$.

Therefore, the instantaneous rate of change is given by the <u>derivative</u> f'.

<u>Example 19:</u> Find the average rate of change in volume of a sphere with respect to its radius r as r changes from 3 to 4. Find the instantaneous rate of change when the radius is 3.

$V = \frac{4}{3}\pi r^3$

$V(r) = \frac{4}{3}\pi r^3$

$V(4) = \frac{4}{3}\pi (4)^3 = \frac{256\pi}{3}$

$V(3) = \frac{4}{3}\pi (3)^3 = \frac{4}{3}\pi (27) = \frac{108\pi}{3}$

Average rate of change $= \frac{V(4) - V(3)}{4 - 3} = \frac{\frac{256\pi}{3} - \frac{108\pi}{3}}{1} = \boxed{\frac{148\pi}{3}}$

Instantaneous rate of change:

$V(r) = \frac{4}{3}\pi r^3 \Rightarrow V'(r) = \frac{4}{3}\pi (3r^2) = 4\pi r^2$

$V'(3) = 4\pi (3)^2 = \boxed{36\pi}$

<u>Example 20:</u> Find the rate of change of the area of a circle with respect to (a) the diameter; (b) the circumference.

Area $= A = \pi r^2$

(a) Put this in terms of diameter:

radius $= \frac{1}{2}$(diameter) $\Rightarrow$ diameter $= 2$(radius)

$r = \frac{1}{2}d \Rightarrow A = \pi\left(\frac{1}{2}d\right)^2$ [using $A = \pi r^2$]

$A(d) = \pi\left(\frac{d^2}{4}\right) = \frac{\pi d^2}{4} = \frac{\pi}{4}(d^2)$

rate of change: $A'(d) = \frac{\pi}{4}(2d) = \boxed{\frac{\pi d}{2}}$

(b) Put $A = \pi r^2$ in terms of circumfence C:

$C = 2\pi r$

$\frac{C}{2\pi} = r$

Substitute this into $A = \pi r^2$:

$A(C) = \pi\left(\frac{C}{2\pi}\right)^2 = \pi\left(\frac{C^2}{4\pi^2}\right) = \frac{C^2}{4\pi} = \frac{1}{4\pi}(C^2)$

$A'(C) = \frac{1}{4\pi}(2C) = \frac{2C}{4\pi} = \boxed{\frac{C}{2\pi}}$

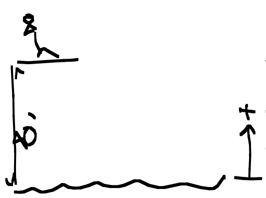
Velocity:

If the independent variable represents *time*, then the derivative can be used to analyze motion.

If the function $s(t)$ represents the position of an object, then the derivative $s'(t) = \frac{ds}{dt}$ is the <u>velocity</u> of the object.

(The velocity is the instantaneous rate of change in distance. The average velocity is the average rate of change in distance.)

<u>Example 21:</u> A person stands on a bridge 40 feet above a river. He throws a ball vertically upward with an initial velocity of 50 ft/sec. Its height (in feet) above the river after t seconds is $s = -16t^2 + 50t + 40$.



a) What is the velocity after 3 seconds?
b) How high will it go?
c) How long will it take to reach a velocity of 20 ft/sec?
d) When will it hit the water? How fast will it be going when it gets there?

$s(t) = -16t^2 + 50t + 40$

(a) $s'(t) = v(t) = \frac{ds}{dt} = -32t + 50$

$v(3) = s'(3) = -32(3) + 50 = -96 + 50 = \boxed{-46 \text{ ft/sec}}$

(b) At max height, velocity $= 0$:

Set $s'(t) = 0$: $-32t + 50 = 0$

$50 = 32t$

$\frac{50}{32} = t$

$t = \frac{25}{16}$ It takes $1\frac{9}{16}$ seconds to reach max height

Max height $= s\left(\frac{25}{16}\right) = -16\left(\frac{25}{16}\right)^2 + 50\left(\frac{25}{16}\right) + 40 = \boxed{79.0625 \text{ ft}}$

(c) Set $v(t) = s'(t) = 20$: $-32t + 50 = 20$

$-32t = -30$

$t = \frac{-30}{-32} = \boxed{\frac{15}{16} \text{ sec}}$

(d) when it hits the water,

$s(t) = 0$: $-16t^2 + 50t + 40 = 0$

$t = \frac{-50 \pm \sqrt{(50)^2 - 4(-16)(40)}}{2(-16)} \Rightarrow \boxed{t \approx 3.785 \text{ sec}}$

$t \approx -0.66$ sec (throw out)

Example 22: Suppose a bullet is shot straight up at an initial velocity of 73 feet per second. If air resistance is neglected, its height from the ground (in feet) after t seconds is given by $h(t) = -16.1t^2 + 73t$.

a. The velocity after 2 seconds.
b. How high will the bullet go?
c. When will the bullet reach the ground?
d. How fast will it be traveling when it hits the ground?

Velocity is $h'(t) = -32.2t + 73$

(a) $h'(2) = -32.2(2) + 73 = \boxed{8.6 \text{ ft/sec}}$

(b) At max height, $h'(t) = 0$:

$$-32.2t + 73 = 0$$
$$-32.2t = -73$$
$$t = \frac{-73}{-32.2} \text{ sec} \approx 2.267 \text{ sec}$$

Max height $= h\left(\frac{73}{32.2}\right) = -16.1\left(\frac{73}{32.2}\right)^2 + 73\left(\frac{73}{32.2}\right) \approx \boxed{82.748 \text{ ft}}$

(c) set $h(t) = 0$:

$-16.1t^2 + 73t = 0$

$t(-16.1t + 73) = 0$

$t = 0$, $\quad -16.1t + 73 = 0$

$-16.1t = -73$

$t = \frac{-73}{-16.1} = \frac{73}{16.1} \approx 4.534$ sec

It hits the ground at approximately 4.534 sec.

(d) $h'\left(\frac{73}{16.1}\right) = -32.2\left(\frac{73}{16.1}\right) + 73$

$= -73$ ft/sec

Speed is 73 ft/sec when it hits ground

Example 23: Suppose the position of a particle is given by $f(t) = t^4 - 32t + 7$. What is the velocity after 3 seconds? When is the particle at rest?

$f'(t) = 4t^3 - 32$

after 3 seconds, velocity is $f'(3) = 4(3)^3 - 32 = \boxed{76 \text{ units/sec}}$

To find when particle is at rest, set $f'(t) = 0$:

$$4t^3 - 32 = 0$$
$$4t^3 = 32$$
$$t^3 = 8$$
$$t = 2 \text{ seconds.}$$

Particle is at rest when $t = 2$ seconds.