2.3: Product and Quotient Rules and Higher Order Derivatives

The Product Rule:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

"the first times the derivative of the second plus the second times the derivative of the first"

Example 1: Find the derivative of $f(x) = x^7(4x^3)$.

Using Product Rule:

$$f'(x) = x^7 \frac{d}{dx}(4x^3) + (4x^3)\frac{d}{dx}(x^7)$$
$$= x^7(12x^2) + (4x^3)(7x^6)$$
$$= 12x^9 + 28x^9 = \boxed{40x^9}$$

The easy way:

$$f(x) = 4x^{10}$$
$$f'(x) = \boxed{40x^9}$$

Example 2: Find the derivative of $f(x) = (4x^3 + x^2 - 2)(x^4 + 8)$.

$$f'(x) = (4x^3 + x^2 - 2)\frac{d}{dx}(x^4 + 8) + (x^4 + 8)\frac{d}{dx}(4x^3 + x^2 - 2)$$
$$= (4x^3 + x^2 - 2)(4x^3) + (x^4 + 8)(12x^2 + 2x).$$

I simplified it after class:

$$= 16x^6 + 4x^5 - 8x^3 + 12x^6 + 2x^5 + 96x^2 + 16x$$
$$= \boxed{28x^6 + 6x^5 - 8x^3 + 96x^2 + 16x}$$

Example 3: Find the derivative of $f(x) = \sqrt{x}(x^5 - 3x^2 + 12x)$.

Example 4: Find the derivative of $(4x^3 + 1)(\sqrt{x} + \frac{1}{x} - 2x)$.

<u>The Quotient Rule</u>:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

<u>Example 5:</u> Find the derivative of $f(x)=\dfrac{-x^5}{4x^3}$.

$$f'(x)=\frac{4x^3\frac{d}{dx}(-x^5)-(-x^5)\frac{d}{dx}(4x^3)}{(4x^3)^2}$$

$$=\frac{4x^3(-5x^4)+x^5(12x^2)}{16x^6}$$

$$=\frac{-20x^7+12x^7}{16x^6}=\frac{-8x^7}{16x^6}=\boxed{-\frac{x}{2}}$$

match

Without quotient rule:

$$f(x)=\frac{-x^5}{4x^3}=-\frac{x^2}{4}$$

$$=-\frac{1}{4}x^2$$

$$f'(x)=-\frac{1}{4}(2x)$$

$$=\boxed{-\frac{x}{2}}$$

<u>Example 6:</u> Find the derivative of $g(x)=\dfrac{4-2x^3}{x^4+3x^7}$.

$$g'(x)=\frac{(x^4+3x^7)\frac{d}{dx}(4-2x^3)-(4-2x^3)\frac{d}{dx}(x^4+3x^7)}{(x^4+3x^7)^2}$$

$$=\frac{(x^4+3x^7)(-6x^2)-(4-2x^3)(4x^3+21x^6)}{(x^4+3x^7)^2}=\frac{-6x^6-18x^9-(16x^3+84x^6-8x^6-42x^9)}{(x^4+3x^7)^2}$$

$$=\frac{-6x^6-18x^9-16x^3-76x^6+42x^9}{(x^4+3x^7)^2}=\boxed{\frac{24x^9-82x^6-16x^3}{(x^4+3x^7)^2}}$$

can simplify more

<u>Example 7:</u> Find the derivative of $f(x)=\dfrac{\sqrt{x}}{x^3-x^4}$.

$$f(x)=\frac{x^{\frac{1}{2}}}{x^3-x^4}$$

$$f'(x)=\frac{(x^3-x^4)\frac{d}{dx}(x^{\frac{1}{2}})-x^{\frac{1}{2}}\frac{d}{dx}(x^3-x^4)}{(x^3-x^4)^2}=\frac{(x^3-x^4)(\frac{1}{2}x^{-\frac{1}{2}})-x^{\frac{1}{2}}(3x^2-4x^3)}{(x^3-x^4)^2}$$

$$=\frac{\frac{1}{2}x^{5/2}-\frac{1}{2}x^{7/2}-3x^{5/2}+4x^{7/2}}{(x^3-x^4)^2}=\boxed{\frac{\frac{7}{2}x^{7/2}-\frac{5}{2}x^{5/2}}{(x^3-x^4)^2}}$$

Example 8: Find the derivative of $f(x) = x^2 \sin x$.

$$f'(x) = x^2 \frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)$$

$$= x^2(\cos x) + (\sin x)(2x)$$

$$= \boxed{x^2 \cos x + 2x \sin x} = \boxed{x(x\cos x + 2\sin x)}$$

Example 9: Find the derivative of $f(x) = x + \sin x \cos x$.

$$f'(x) = 1 + (\sin x)\frac{d}{dx}(\cos x) + (\cos x)\left(\frac{d}{dx}(\sin x)\right)$$

$$= 1 + (\sin x)(-\sin x) + (\cos x)(\cos x)$$

$$= \underbrace{1 - \sin^2 x}_{\cos^2 x} + \cos^2 x$$

$$= \cos^2 x + \cos^2 x = \boxed{2\cos^2 x}$$

Recall: Pythagorean Identities

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$\sin^2\theta = 1 - \cos^2\theta$

OR

$$f'(x) = 1 - \sin^2 x + \cos^2 x$$

$$= 1 - \sin^2 x + (1 - \sin^2 x)$$

$$= 2 - 2\sin^2 x$$

Example 10: Find the derivative of $y = \dfrac{1-\cos x}{\sin x}$.

$$\frac{dy}{dx} = \frac{(\sin x)\frac{d}{dx}(1-\cos x) - (1-\cos x)\frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$= \frac{(\sin x)(0 - (-\sin x)) - (1-\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{(\sin x)(\sin x) - (\cos x - \cos^2 x)}{\sin^2 x} = \frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{\overbrace{\sin^2 x + \cos^2 x} - \cos x}{\sin^2 x} = \boxed{\frac{1-\cos x}{\sin^2 x}}$$

Example 11: Prove that $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

(Use the fact that $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\sin x) = \cos x$, along with the quotient rule)

(Not done during class)

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}(1) - 1\frac{d}{dx}(\sin x)}{(\sin x)^2} = \frac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin x}\left(\frac{1}{\sin x}\right) = -\cot x \csc x = -\csc x \cot x$$

////

Higher order derivatives:

Once the derivative of $f(x)$ if also a function, it is possible to find the derivative of $f'(x)$ too. This is called the *second derivative* and is denoted $f''(x)$. The second derivative gives the instantaneous rate of change of the derivative. In other words, it tells us how fast the slope is changing.

Similarly, the derivative of $f''(x)$ can be calculated and this is called the *third derivative* $f'''(x)$.

In general, we can keep calculating the derivative of the previous derivative. The *nth derivative* is found by taking the derivative *n* times. The *nth derivative of f* is denoted $f^{(n)}(x)$.

Other notation: $y', y'', y''', \ldots, y^{(n)}$

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$$

$$D'y, D^2y, D^3y, \ldots, D^ny$$

Example 12: Suppose that $y = -7x^5 + 6x^4 - \frac{2}{x}$. Find y'''.

$$y = -7x^5 + 6x^4 - 2x^{-1}$$

$$y' = -35x^4 + 24x^3 + 2x^{-2}$$

$$y'' = -140x^3 + 72x^2 - 4x^{-3}$$

$$\boxed{y''' = -420x^2 + 144x + 12x^{-4}} = \boxed{-420x^2 + 144x + \frac{12}{x^4}}$$

Example 13: Suppose that $f(x) = \sqrt[3]{x}$. Find the second and third derivatives.

$$f(x) = x^{1/3}$$

$$f'(x) = \frac{1}{3}x^{-2/3}$$

$$f''(x) = \frac{1}{3}\left(-\frac{2}{3}x^{-5/3}\right) = \boxed{-\frac{2}{9}x^{-5/3}}$$

$$f'''(x) = -\frac{2}{9}\left(-\frac{5}{3}x^{-8/3}\right) = \boxed{\frac{10}{27}x^{-8/3}} = \frac{10}{27\sqrt[3]{x^8}}$$

Using derivatives to describe the motion of an object:

If the dependent variable t represents time, and the function $s(t)$ represents the position (distance from a particular point) of an object, then

- the velocity $v(t)$ is the first derivative $s'(t) = \frac{ds}{dt}$.
- the acceleration $a(t)$ is the second derivative $s''(t) = \frac{dv}{dt}$.
- the jerk $j(t)$ is the third derivative $s'''(t) = \frac{da}{dt}$.
- the speed is the absolute value of the velocity $|v(t)| = \left|\frac{ds}{dt}\right|$.

Example 14: The position (in feet) of an object is given by $s(t) = t^4 - 32t + 7$, with t measured in seconds. Find functions representing the velocity, acceleration, and jerk. Find the velocity and acceleration after 1, 2, and 3 seconds.

$s(t) = t^4 - 32t + 7$

Velocity: $v(t) = s'(t) = 4t^3 - 32$

acceleration: $a(t) = v'(t) = s''(t) = 12t^2$

jerk: $j(t) = a'(t) = s'''(t) = 24t$

$v(1) = 4(1)^3 - 32 = 4 - 32 = \boxed{-28 \text{ ft/s}}$

$a(1) = 12(1)^2 = \boxed{12 \text{ ft/s}^2}$

acceleration $= \frac{dv}{dt} \Rightarrow \frac{\text{ft/s}}{\text{sec}} = \frac{\text{ft}}{\text{sec}} \cdot \frac{1}{\text{sec}} = \text{ft/sec}^2$