2.4: The Chain Rule

The chain rule allows us to find the derivative of many more functions.

Example 1: Suppose $g(x) = (x^3 + 2)^2$. Find $g'(x)$.

$g(x) = x^6 + 4x^3 + 4$

$\boxed{g'(x) = 6x^5 + 12x^2}$

What if we had
$g(x) = (x^3+2)^{12}$?

The chain rule lets us differentiate composite functions.

The Chain Rule:

If f and g are both differentiable and $F(x) = f(g(x))$, then F is differentiable and

$$F'(x) = f'(g(x))\, g'(x).$$

Alternatively, if $y = f(u)$ and $u = f(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

With the chain rule, we take the derivative of the "outer function" and multiply by the derivative of the "inner function".

Example 2: Find the derivative of $h(x) = (x^3 + 2)^{50}$.

$h'(x) = 50(x^3+2)^{49} \frac{d}{dx}(x^3+2)$

$= 50(x^3+2)^{49}(3x^2)$

$= \boxed{150x^2(x^3+2)^{49}}$

Ex $2\frac{1}{2}$:

$f(x) = \cos(x^3+2)$

$f'(x) = [-\sin(x^3+2)] \frac{d}{dx}(x^3+2)$

$= [-\sin(x^3+2)](3x^2)$

$= \boxed{-3x^2 \sin(x^3+2)}$

Example 3: Find the derivative of $f(x)=\sqrt{2x^3-5x}$.

$$f(x) = (2x^3-5x)^{1/2}$$
$$f'(x) = \frac{1}{2}(2x^3-5x)^{-\frac{1}{2}}\frac{d}{dx}(2x^3-5x)$$
$$= \frac{1}{2}(2x^3-5x)^{-1/2}(6x^2-5) = \boxed{\frac{6x^2-5}{2\sqrt{2x^3-5x}}}$$

Example 4: Suppose that $y=\dfrac{2}{(3x^5-4)^3}$. Find $\dfrac{dy}{dx}$.

$$y = 2(3x^5-4)^{-3}$$
$$\frac{dy}{dx} = -6(3x^5-4)^{-4}\frac{d}{dx}(3x^5-4)$$
$$= -6(3x^5-4)^{-4}(15x^4) = \boxed{-\frac{90x^4}{(3x^5-4)^4}}$$

Example 5: Find the derivative of $f(x)=\sqrt[3]{\cos x}$.

$$f(x) = (\cos x)^{1/3}$$
$$f'(x) = \frac{1}{3}(\cos x)^{-2/3}\frac{d}{dx}(\cos x)$$
$$= \frac{1}{3}(\cos x)^{-2/3}(-\sin x) = \boxed{-\frac{\sin x}{3\sqrt[3]{\cos^2 x}}}$$

Example 6: Find the derivative of $g(x)=\cos\sqrt[3]{x}$.

$$g(x) = \cos(\sqrt[3]{x}) = \cos(x^{1/3})$$
$$g'(x) = -[\sin(x^{1/3})]\frac{d}{dx}(x^{1/3}) = -[\sin(x^{1/3})]\left(\frac{1}{3}x^{-2/3}\right)$$
$$= -\frac{1}{3}x^{-2/3}\sin(x^{1/3})$$
$$= \boxed{-\frac{\sin(\sqrt[3]{x})}{3\sqrt[3]{x^2}}}$$

Example 7: Suppose that $y=\dfrac{1}{\cos x}$. Find $\dfrac{dy}{dx}$.

$$y = (\cos x)^{-1}$$
$$\frac{dy}{dx} = -1(\cos x)^{-2}\frac{d}{dx}(\cos x)$$
$$= -1(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$$
$$= \tan x \sec x$$
$$= \boxed{\sec x \tan x}$$

Recall:

$$\frac{d}{dx}(\sec x) = \sec x\tan x$$

Example 8: Suppose that $h(x)=(3x^2-4)^3(2x-9)^2$. Find $h'(x)$.

$h'(x) = (3x^2-4)^3 \frac{d}{dx}(2x-9)^2 + (2x-9)^2 \frac{d}{dx}(3x^2-4)^3 \leftarrow$ Product Rule

$= (3x^2-4)^3(2)(2x-9)\frac{d}{dx}(2x-9) + (2x-9)^2(3)(3x^2-4)^2\frac{d}{dx}(3x^2-4)$

$= (3x^2-4)^3(2)(2x-9)(2) + (2x-9)^2(3)(3x^2-4)^2(6x)$

$= \underline{4(3x^2-4)^3(2x-9)} + \underline{18x(2x-9)^2(3x^2-4)^2}$

$= 2(3x^2-4)^2(2x-9)\left[2(3x^2-4)+9x(2x-9)\right]$

$= 2(3x^2-4)^2(2x-9)(6x^2-8+18x^2-81x)$

$= \boxed{2(3x^2-4)^2(2x-9)(24x^2-81x-8)}$

Example 9: Suppose that $f(x)=\left(\frac{2x+1}{2x-1}\right)^5$. Find $f'(x)$.

$f'(x) = 5\left(\frac{2x+1}{2x-1}\right)^4 \frac{d}{dx}\left(\frac{2x+1}{2x-1}\right)$

$= 5\left(\frac{2x+1}{2x-1}\right)^4 \frac{(2x-1)\frac{d}{dx}(2x+1)-(2x+1)\frac{d}{dx}(2x-1)}{(2x-1)^2}$

$= 5\left(\frac{2x+1}{2x-1}\right)^4 \frac{(2x-1)(2)-(2x+1)(2)}{(2x-1)^2} = 5\left(\frac{2x+1}{2x-1}\right)^4 \frac{\cancel{4x}-2-\cancel{4x}-2}{(2x-1)^2}$

$= 5\left(\frac{2x+1}{2x-1}\right)^4\left(\frac{-4}{(2x-1)^2}\right) = \boxed{-\frac{20(2x+1)^4}{(2x-1)^6}}$

Example 10: Find the derivative of $y=\cos(\sin(\pi x^2))$.

$\frac{dy}{dx} = -\sin\left(\sin(\pi x^2)\right)\frac{d}{dx}\left(\sin(\pi x^2)\right)$

$= -\sin\left(\sin(\pi x^2)\right)\cos(\pi x^2)\frac{d}{dx}(\pi x^2)$

$= -\sin\left(\sin(\pi x^2)\right)\left(\cos(\pi x^2)\right)(2\pi x)$

Example 11: Find the derivative of $g(x)=\dfrac{\cos^2 x}{\sqrt{4x+1}}$ $\quad g(x)=\dfrac{(\cos x)^2}{(4x+1)^{1/2}}$

$$g'(x)=\frac{(4x+1)^{1/2}\frac{d}{dx}(\cos x)^2-(\cos x)^2\frac{d}{dx}(4x+1)^{1/2}}{\left[(4x+1)^{1/2}\right]^2}$$

$$=\frac{(4x+1)^{1/2}(2)(\cos x)(-\sin x)-(\cos x)^2\left(\tfrac{1}{2}\right)(4x+1)^{-1/2}(4)}{4x+1}$$

$$=\frac{-2\cos x\sin x\sqrt{4x+1}-\dfrac{2\cancel{4}(\cos x)^2}{\cancel{2}\sqrt{4x+1}}\cdot\dfrac{\sqrt{4x+1}}{\sqrt{4x+1}}}{4x+1}=\frac{-2(\cos x\sin x)(4x+1)-2\cos^2 x}{(4x+1)^{3/2}}$$

Example 12: Find the first and second derivatives of $f(x)=(x^2+4)^5$.

$$f'(x)=5(x^2+4)^4(2x)=\boxed{10x(x^2+4)^4}$$

$$f''(x)=10x\frac{d}{dx}(x^2+4)^4+(x^2+4)^4\frac{d}{dx}(10x)$$

$$=10x(4)(x^2+4)^3(2x)+(x^2+4)^4(10)$$

$$=80x^2(x^2+4)^3+10(x^2+4)^4$$

$$=10(x^2+4)^3\left[8x^2+x^2+4\right]=\boxed{10(x^2+4)^3(9x^2+4)}$$

Example 13: Find the first and second derivatives of $f(x)=\cos\left(3x^2\right)$.

$$f'(x)=-\left(\sin(3x^2)\right)(6x)=\boxed{\underbrace{-6x}_{1^{st}}\underbrace{\sin(3x^2)}_{2^{nd}}}$$

$$f''(x)=-6x\frac{d}{dx}(\sin(3x^2))+\sin(3x^2)\frac{d}{dx}(-6x)$$

$$=-6x(\cos(3x^2))(6x)+(\sin(3x^2))(-6)$$

$$=\boxed{-36x^2\cos(3x^2)-6\sin(3x^2)}$$

Example 14: Find the first and second derivatives of $y = \dfrac{2}{(3x+5)^2}$

$$y = 2(3x+5)^{-2}$$

$$\frac{dy}{dx} = y' = -4(3x+5)^{-3}\frac{d}{dx}(3x+5) = -4(3x+5)^{-3}(3) = \boxed{-12(3x+5)^{-3}}$$

$$= \boxed{-\frac{12}{(3x+5)^3}}$$

$$\frac{d^2y}{dx^2} = y'' = 36(3x+5)^{-4}(3)$$

$$= \boxed{108(3x+5)^{-4}} = \boxed{\frac{108}{(3x+5)^4}}$$

Example 15: Find the equation of the tangent line to $y = \dfrac{x^2}{\sin \pi x + 1}$ at the point where $x = 1$.

$$y = \frac{x^2}{\sin(\pi x) + 1}$$

$$\frac{dy}{dx} = \frac{(\sin(\pi x)+1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(\sin(\pi x)+1)}{(\sin \pi x + 1)^2}$$

$$= \frac{(\sin(\pi x)+1)(2x) - x^2((\cos \pi x)(\pi) + 0)}{(\sin(\pi x)+1)^2}$$

$$= \frac{2x(\sin(\pi x)+1) - \pi x^2 \cos(\pi x)}{(\sin(\pi x)+1)^2}$$

Note:

$$\frac{d}{dx}(\sin(\pi x) + 1$$
$$= \cos(\pi x)\frac{d}{dx}(\pi x)$$
$$= \boxed{\cos(\pi x)}(\pi)$$
$$= \pi\cos(\pi x)$$

$$m = \left.\frac{dy}{dx}\right|_{x=1} = \frac{2(1)(\sin \pi + 1) - \pi(1)^2\cos\pi}{(\sin\pi + 1)^2}$$

$$= \frac{2(0+1) - \pi(-1)}{(0+1)^2} = \frac{2+\pi}{1} = 2+\pi$$

Find y-value: $\left. y\right|_{x=1} = \left.\dfrac{x^2}{\sin \pi x + 1}\right|_{x=1} = \dfrac{1^2}{\sin\pi + 1} = \dfrac{1}{0+1} = 1$

$m = 2+\pi$, Point: $(1,1)$

$$y - y_1 = m(x - x_1) \Rightarrow y - 1 = (2+\pi)(x-1) \Rightarrow y - 1 = 2x - 2 + \pi x - \pi$$

$$\boxed{y = 2x + \pi x - 1 - \pi}$$

$$y = (2+\pi)x - 1 - \pi$$