2.5: Implicit Differentiation

Note: $\frac{d}{dx}(y) = \frac{dy}{dx}$

Example 1: Given the equation $x^3 - 4y - 9x^2 = 5$, find $\frac{dy}{dx}$ by

a) Solving explicitly for y.
b) Implicit differentiation.

(a) Solve for y:

$$x^3 - 9x^2 - 5 = 4y$$
$$\frac{x^3 - 9x^2 - 5}{4} = y$$
$$y = \frac{1}{4}(x^3 - 9x^2 - 5)$$
$$\frac{dy}{dx} = \frac{1}{4}(3x^2 - 18x) = \boxed{\frac{3}{4}x^2 - \frac{9}{2}x}$$

(b)

$$x^3 - 4y - 9x^2 = 5$$
$$\frac{d}{dx}(x^3 - 4y - 9x^2) = \frac{d}{dx}(5)$$
$$3x^2 - 4\frac{dy}{dx} - 18x = 0$$

Solve for $\frac{dy}{dx}$:

$$-4\frac{dy}{dx} = -3x^2 + 18x$$
$$\frac{-4\frac{dy}{dx}}{-4} = \frac{-3x^2}{-4} + \frac{18x}{-4}$$
$$\boxed{\frac{dy}{dx} = \frac{3}{4}x^2 - \frac{9}{2}x}$$

Example 2: Find $\frac{dy}{dx}$ for $xy = 4$.

$$\frac{d}{dx}(xy) = \frac{d}{dx}(4)$$

need product rule

$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = 0$$
$$x\frac{dy}{dx} + y(1) = 0$$
$$x\frac{dy}{dx} = -y$$
$$\boxed{\frac{dy}{dx} = -\frac{y}{x}}$$

Example 3: Find $\frac{dy}{dx}$ for the equation $x^3y - 2x^2y^3 + x^2 - 3 = 0$.

$$\frac{d}{dx}(x^3y - 2\underbrace{x^2}_{1st}\underbrace{y^3}_{2nd} + x^2 - 3) = \frac{d}{dx}(0)$$

[Product Rule again! on 1st 2 terms]

$$x^3\frac{d}{dx}(y) + y\frac{d}{dx}(x^3) - 2\left(x^2\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(x^2)\right) + 2x - 0 = 0$$
$$x^3\frac{dy}{dx} + y(3x^2) - 2\left(x^2(3y^2)\frac{dy}{dx} + y^3(2x)\right) + 2x = 0$$
$$x^3\frac{dy}{dx} + 3x^2y - 6x^2y^2\frac{dy}{dx} - 4xy^3 + 2x = 0$$

To solve for $\frac{dy}{dx}$, put all terms with $\frac{dy}{dx}$ on 1 side; put terms without $\frac{dy}{dx}$ on other side.

$$x^3\frac{dy}{dx} - 6x^2y^2\frac{dy}{dx} = -3x^2y + 4xy^3 - 2x$$
$$\frac{dy}{dx}(x^3 - 6x^2y^2) = -3x^2y + 4xy^3 - 2x \Rightarrow \boxed{\frac{dy}{dx} = \frac{-3x^2y + 4xy^3 - 2x}{x^3 - 6x^2y^2}}$$

could factor out an x and reduce

Ex $1\frac{1}{2}$ Find $\frac{d}{dx}(y^3)$.

$$= 3(y)^2 \frac{d}{dx}(y)$$

$$= 3y^2 \frac{dy}{dx}$$

$$\frac{d}{dx}(x^2+7)^3$$

$$= 3(x^2+7)^2(2x)$$

Ex $1\frac{6}{10}$

$$\frac{d}{dx}(y^5)$$

$$= 5y^4 \frac{dy}{dx}$$

Ex $1\frac{7}{10}$

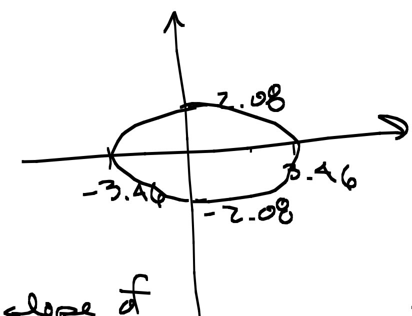
$$\frac{d}{dx}(\cos(y)) = -(\sin(y))\frac{dy}{dx}$$

Ex $1\frac{8}{10}$ Find $\frac{dy}{dx}$ for $x^2 + 9y^2 = 12$

It's an ellipse:

$$\frac{x^2}{12} + \frac{9y^2}{12} = \frac{12}{12}$$

$$\frac{x^2}{12} + \frac{y^2}{4/3} = 1$$



$$\frac{d}{dx}(x^2+9y^2) = \frac{d}{dx}(12)$$

$$2x + 18y\frac{dy}{dx} = 0$$

$$18y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{18y}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{9y}}$$

This is the slope of the tangent line at the point (x, y).

Solving explicitly for y would be messier.

$$9y^2 = 12 - x^2$$

$$y^2 = \frac{12}{9} - \frac{x^2}{9}$$

$$y = \pm\sqrt{\frac{12}{9} - \frac{x^2}{9}}$$

Example 4: Find $\frac{dy}{dx}$ for the equation $\frac{1}{y^4} - \frac{5}{x^4} = 1$.

Rewrite: $y^{-4} - 5x^{-4} = 1$

$$\frac{d}{dx}\left(y^{-4} - 5x^{-4}\right) = \frac{d}{dx}(1)$$

$$-4y^{-5}\frac{dy}{dx} + 20x^{-5} = 0$$

$$-\frac{4}{y^5}\frac{dy}{dx} + \frac{20}{x^5} = 0$$

$$-\frac{4}{y^5}\frac{dy}{dx} = -\frac{20}{x^5}$$

$$\left(-\frac{y^5}{4}\right)\left(-\frac{4}{y^5}\right)\frac{dy}{dx} = -\frac{20}{x^5}\left(-\frac{y^5}{4}\right)$$

$$\frac{dy}{dx} = \frac{20y^5}{4x^5}$$

$$\boxed{\frac{dy}{dx} = \frac{5y^5}{x^5}}$$

Example 5: Find $\frac{dy}{dx}$ for the equation $(x-y)^4 = y^2$.

$$\frac{d}{dx}(x-y)^4 = \frac{d}{dx}(y^2)$$

$$4(x-y)^3\frac{d}{dx}(x-y) = 2y\frac{dy}{dx}$$

$$4(x-y)^3\left(1-\frac{dy}{dx}\right) = 2y\frac{dy}{dx}$$

$$4(x-y)^3 - 4(x-y)^3\frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$4(x-y)^3 = 4(x-y)^3\frac{dy}{dx} + 2y\frac{dy}{dx}$$

Divide by 2:

$$2(x-y)^3 = 2(x-y)^3\frac{dy}{dx} + y\frac{dy}{dx}$$

$$2(x-y)^3 = \frac{dy}{dx}\left(2(x-y)^3 + y\right)$$

$$\boxed{\frac{2(x-y)^3}{2(x-y)^3 + y} = \frac{dy}{dx}}$$

Example 6: Find $\frac{dy}{dx}$ for the equation $x + \cos(x^2 y) = y$

$$\frac{d}{dx}\left(x + \cos(x^2y)\right) = \frac{d}{dx}(y)$$

$$1 - \left(\sin(x^2y)\right)\frac{d}{dx}(x^2y) = \frac{dy}{dx}$$

$$1 - \left(\sin(x^2y)\right)\underbrace{\left(x^2\frac{dy}{dx} + y(2x)\right)}_{\text{product rule}} = \frac{dy}{dx}$$

$$1 - x^2\frac{dy}{dx}\sin(x^2y) - 2xy\sin(x^2y) = \frac{dy}{dx}$$

$$1 - 2xy\sin(x^2y) = \frac{dy}{dx} + x^2\frac{dy}{dx}\sin(x^2y)$$

$$1 - 2xy\sin(x^2y) = \frac{dy}{dx}\left(1 + x^2\sin(x^2y)\right)$$

$$\boxed{\frac{1-2xy\sin(x^2y)}{1+x^2\sin(x^2y)} = \frac{dy}{dx}}$$

Note: This is equivalent to:

$$\boxed{\frac{dy}{dx} = \frac{-1+2xy\sin(x^2y)}{-1-x^2\sin(x^2y)}}$$

<u>Finding higher-order derivatives using implicit differentiation</u>:

To find the second derivative, denoted $\frac{d^2y}{dx^2}$, differentiate the first derivative $\frac{dy}{dx}$ with respect to *x*.

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

<u>Example 7:</u> Find $\frac{d^2y}{dx^2}$ for the equation $x^3 - 2x^2 = y$.

$$\frac{d}{dx}(x^3 - 2x^2) = \frac{d}{dx}(y)$$

$$3x^2 - 4x = \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 4x)$$

$$= \boxed{6x - 4}$$

<u>Example 8:</u> Find $\frac{d^2y}{dx^2}$ for the equation $xy^2 - y = 3$.

$$\frac{d}{dx}(xy^2 - y) = \frac{d}{dx}(3)$$

$$x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x) - \frac{dy}{dx} = 0$$

$$x \cdot 2y\frac{dy}{dx} + y^2(1) - \frac{dy}{dx} = 0$$

$$2xy\frac{dy}{dx} - \frac{dy}{dx} = -y^2$$

$$\frac{dy}{dx}(2xy - 1) = -y^2$$

$$\frac{dy}{dx} = \frac{-y^2}{2xy-1}$$

$$\frac{d^2y}{dx^2} = \frac{(2xy-1)\frac{d}{dx}(-y^2) - (-y^2)\frac{d}{dx}(2xy-1)}{(2xy-1)^2}$$

$$= \frac{(2xy-1)\left(-2y\frac{dy}{dx}\right) + y^2\left(2x\frac{dy}{dx} + y(2)\right)}{(2xy-1)^2}$$

$$= \frac{-2y\frac{dy}{dx}(2xy-1) + 2xy^2\frac{dy}{dx} + 2y^3}{(2xy-1)^2}$$

Put in $\frac{dy}{dx} = \frac{-y^2}{2xy-1}$

$$= \frac{-2y\left(-\frac{y^2}{\cancel{2xy-1}}\right)\cancel{(2xy-1)} + 2xy^2\left(\frac{-y^2}{2xy-1}\right) + 2y^3}{(2xy-1)^2}$$

See next page

Ex 8 cont'd

$$= \frac{2y^3 - \frac{2xy^4}{2xy-1} + 2y^3}{(2xy-1)^2} \cdot \frac{2xy-1}{2xy-1}$$

$$= \frac{2y^3(2xy-1) - 2xy^4 + 2y^3(2xy-1)}{(2xy-1)^3}$$

$$= \frac{4xy^4 - 2y^3 - 2xy^4 + 4xy^4 - 2y^3}{(2xy-1)^3}$$

$$= \boxed{\frac{6xy^4 - 4y^3}{(2xy-1)^3}}$$

Example 9: Find the equation of the tangent line to the ellipse $4x^2+16y^2=64$ at the point $(2,\sqrt{3})$.

Check: is $(2,\sqrt{3})$ on the curve?

$$4(2)^2+16(\sqrt{3})^2=64$$
$$16+48=64$$
$$64=64\ \checkmark$$

Find $\frac{dy}{dx}$

$$\frac{d}{dx}(4x^2+16y^2)=\frac{d}{dx}(64)$$
$$8x+32y\frac{dy}{dx}=0$$
$$32y\frac{dy}{dx}=-8x$$

$$\frac{dy}{dx}=\frac{-8x}{32y}=-\frac{x}{4y}$$

$$m=\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=\sqrt{3}}}=\left.-\frac{x}{4y}\right|_{\substack{x=2\\y=\sqrt{3}}}$$
$$=-\frac{2}{4\sqrt{3}}=-\frac{1}{2\sqrt{3}}=-\frac{\sqrt{3}}{6}$$

Write eqn:

$$y-y_1=m(x-x_1)$$
$$y-\sqrt{3}=-\frac{\sqrt{3}}{6}(x-2)$$
$$y=-\frac{\sqrt{3}}{6}x+\frac{2\sqrt{3}}{6}+\sqrt{3}$$
$$y=-\frac{\sqrt{3}}{6}x+\frac{\sqrt{3}}{3}+\frac{3\sqrt{3}}{3}$$
$$\boxed{y=-\frac{\sqrt{3}}{6}x+\frac{4\sqrt{3}}{3}}$$

Definition: Two curves are said to be *orthogonal* if at each point of intersection, their tangent lines are perpendicular.

Example 10: Show that the hyperbola $x^2-y^2=5$ and the ellipse $4x^2+9y^2=72$ are orthogonal.

Find the intersection points:

$$x^2-y^2=5 \Rightarrow x^2=y^2+5$$

Put $x^2=y^2+5$ into $4x^2+9y^2=72$

$$4(y^2+5)+9y^2=72$$
$$4y^2+20+9y^2=72$$
$$13y^2=52$$
$$y^2=4 \Rightarrow y=\pm 2$$

Back to $x^2=y^2+5$:

$y=2$: $x^2=2^2+5$, $x^2=9$, $x=\pm 3$

$y=-2$: $x^2=(-2)^2+5$, $x^2=4+5$, $x^2=9$, $x=\pm 3$

So, intersection points are $(3,2)$, $(-3,2)$, $(3,-2)$, $(-3,-2)$

Differentiate each curve:

$$\frac{d}{dx}(x^2-y^2)=\frac{d}{dx}(5)$$
$$2x-2y\frac{dy}{dx}=0$$
$$2x=2y\frac{dy}{dx}$$
$$m_1=\frac{dy}{dx}=\frac{2x}{2y}=\frac{x}{y}$$

2nd curve: $\frac{d}{dx}(4x^2+9y^2)=\frac{d}{dx}(72)$

$$8x+18y\frac{dy}{dx}=0$$
$$18y\frac{dy}{dx}=-8x$$
$$m_2=\frac{dy}{dx}=-\frac{8x}{18y}=-\frac{4x}{9y}$$

At $(3,2)$: $m_1=\frac{x}{y}=\frac{3}{2}$

$$m_2=-\frac{4x}{9y}=-\frac{4(3)}{9(2)}=-\frac{12}{18}=-\frac{2}{3}$$

compare

Slopes are opposite reciprocals, so tangent lines are perpendicular at $(3,2)$. Repeat for other 3 points.

Homework Questions (Mon 2/23)

2.4 #23 $f(x) = x^2 (x-2)^4$

Product Rule: $f'(x) = x^2 \frac{d}{dx}(x-2)^4 + (x-2)^4 \frac{d}{dx}(x^2)$

$= x^2(4)(x-2)^3(1) + (x-2)^4(2x)$

$= \underline{4x^2(x-2)^3} + \underline{2x(x-2)^4}$

$= 2x(x-2)^3[2x + (x-2)]$

$= \boxed{2x(x-2)^3(3x-2)}$

2.4 #49 $h(x) = \sin(2x)\cos(2x)$

Recall:
$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos^2\theta = 1-\sin^2\theta$
$\sin^2\theta = 1-\cos^2\theta$

Product Rule: $h'(x) = \sin(2x)\frac{d}{dx}(\cos(2x)) + \cos(2x)\frac{d}{dx}(\sin 2x)$

$= (\sin 2x)(-\sin(2x))(2) + \cos(2x)(\cos(2x))(2)$

$= \boxed{-2\sin^2(2x) + 2\cos^2(2x)}$

$= -2\sin^2(2x) + 2(1-\sin^2(2x))$ ← $\cos^2\theta = 1-\sin^2\theta$

$= -2\sin^2(2x) + 2 - 2\sin^2(2x)$

$= \boxed{2-4\sin^2(2x)}$ could write in terms of $\cos 2x$ instead.

OR, could use a double angle formula and avoid product rule altogether:

$h(x) = \sin(2x)\cos(2x)$

$= \frac{1}{2}\sin(4x)$

$\begin{cases} \sin 2\theta = 2\sin\theta\cos\theta \\ \frac{1}{2}\sin 2\theta = \sin\theta\cos\theta \end{cases}$ ☆

$h'(x) = \frac{1}{2}(\cos(4x))(4)$

$= \boxed{2\cos(4x)}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ ☆

$= 1 - 2\sin^2\theta$

$= 2\cos^2\theta - 1$

replace $\cos^2\theta = 1-\sin^2\theta$

replace $\sin^2\theta = 1-\cos^2\theta$

Make $2-4\sin^2(2x)$ look like this:

$2-4\sin^2(2x)$

$= 2(1-2\sin^2 2x)$

$= 2\cos(2\cdot 2x) = \boxed{2\cos(4x)}$

2.4 #51 $f(x) = \dfrac{\cot x}{\sin x}$

$$= \frac{\frac{\cos x}{\sin x}}{\sin x} = \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = \frac{\cos x}{\sin^2 x}$$

$$f'(x) = \frac{\sin^2 x \frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(\sin x)^2}{(\sin^2 x)^2}$$

$$= \frac{\sin^2 x(-\sin x) - (\cos x)(2)(\sin x)\frac{d}{dx}(\sin x)}{\sin^4 x}$$

$$= \frac{-\sin^3 x - 2\cos x \sin x \cos x}{\sin^4 x} = \frac{-\sin^3 x - 2\cos^2 x \sin x}{\sin^4 x}$$

$$= \frac{\sin x(-\sin^2 x - 2\cos^2 x)}{\sin^4 x} = \frac{-\sin^2 x - 2\cos^2 x}{\sin^3 x}$$

$$= \frac{-\sin^2 x - 2(1 - \sin^2 x)}{\sin^3 x} = \frac{-\sin^2 x - 2 + 2\sin^2 x}{\sin^3 x}$$

$$= \boxed{\frac{-2 + \sin^2 x}{\sin^3 x}}$$

2.4 #83 $f(x) = 2\cos x + \sin 2x$ where on $(0, 2\pi)$ is tangent line horizontal?

$$f'(x) = -2\sin x + (\cos 2x)(2)$$
$$= -2\sin x + 2\cos 2x$$

Recall
$\cos 2\theta = \cos^2\theta - \sin^2\theta$

OR use
$\cos 2\theta = 1 - 2\sin^2\theta$

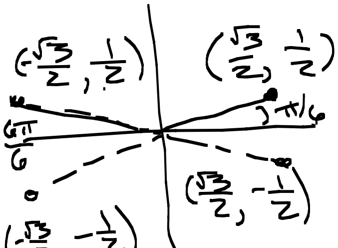
Set equal to 0: $0 = -2\sin x + 2\cos 2x$

$$0 = -2\sin x + 2(\cos^2 x - \sin^2 x) \triangle$$

$$0 = -2\sin x + 2\cos^2 x - 2\sin^2 x$$

$$0 = -2\sin x + 2(1 - \sin^2 x) - 2\sin^2 x$$

$\triangle$ $\cos^2 x = 1 - \sin^2 x$

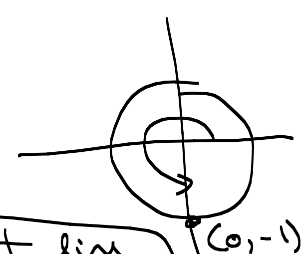


$$0 = -2\sin x + 2 - 4\sin^2 x$$
$$0 = -4\sin^2 x - 2\sin x + 2$$
$$0 = -2(2\sin^2 x + \sin x - 1)$$
$$0 = -2(2\sin x - 1)(\sin x + 1)$$

$2\sin x = 1$ | $\sin x = -1$

$\sin x = \frac{1}{2}$ | $x = \frac{3\pi}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$



Tangent line is horizontal at $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

2.4 # 89

$f(x) = \sin(x^2)$

$f'(x) = (\cos(x^2))(2x)$

$= 2x\cos(x^2)$

$f''(x) = 2x \frac{d}{dx}(\cos(x^2)) + (\cos(x^2))\frac{d}{dx}(2x)$

$= 2x(-\sin(x^2))(2x) + (\cos(x^2))(2)$

$= \boxed{-4x^2\sin(x^2) + 2\cos(x^2)}$