2.6: Related rates

General idea for solving rate problems:

1. Draw a sketch if applicable. The only dimensions you put on your sketch should be those that do not change.
2. Write down, in calculus notation, the rates you know and want.
3. Write an equation relating the quantities that are changing.
4. Differentiate it implicitly, with respect to time.
5. Substitute known quantities.
6. Solve for the required rate.

Example 1: The radius of a sphere is increasing at the rate of 2 inches per minute. Find the rate of change in volume when the radius is 6 inches.

r = radius
V = volume

Know: $\frac{dr}{dt} = +2 \text{ in/min}$

Want: $\frac{dV}{dt}$

When: $r = 6 \text{ in}$

Write an equation that relates the quantity in want to the quantity in know:

$$V = \frac{4}{3}\pi r^3$$

Implicit diff:

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi\left(\cancel{3}r^2 \frac{dr}{dt}\right)$$

$$= 4\pi r^2 \frac{dr}{dt}$$

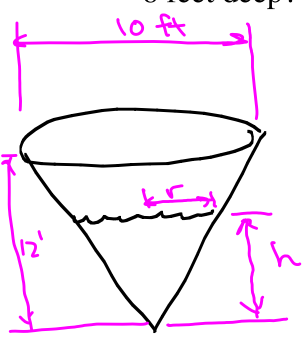
$$\left.\frac{dV}{dt}\right|_{\substack{r=6\text{in} \\ \frac{dr}{dt} = 2\text{in/min}}}$$

$$= 4\pi (6\text{in})^2\left(\frac{2\text{in}}{\text{min}}\right)$$

$$= 8\pi(36)\frac{\text{in}^3}{\text{min}} = \boxed{\frac{288\pi \text{ in}^3}{\text{min}}}$$

$$= \boxed{288\pi \frac{\text{in}^3}{\text{min}}}$$

Example 2: A conical tank (with vertex down) has a diameter of 10 feet at the top and is 12 feet deep. If water is draining out at 10 cubic feet per minute, what is the rate of change in depth when the water is 8 feet deep?



V = volume

Know: $\frac{dV}{dt} = -10 \text{ ft}^3/\text{min}$

Want: $\frac{dh}{dt}$

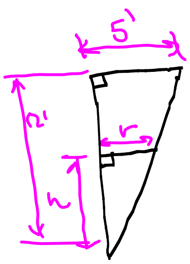
When: $h = 8 \text{ ft}$

Need a formula relating h to V:

Volume of a cone $V = \frac{1}{3}\pi r^2 h$

We want V and h only ... need to get rid of r.

Find an equation that relates r to h.



Similar triangles: $\frac{r}{h} = \frac{5}{12}$ or $\frac{r}{5} = \frac{h}{12}$

Solve for r: $r = \frac{5}{12}h$ $\qquad r = \frac{5h}{12}$

Put this into $V = \frac{1}{3}\pi r^2 h$ in place of r.

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$$V_{cone}: \quad V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{5h}{12}\right)^2 h$$

$$= \frac{1}{3}\pi\left(\frac{25h^2}{144}\right)h$$

$$V = \frac{25\pi h^3}{432}$$

$$V = \frac{25\pi}{432} h^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{25\pi}{432}h^3\right)$$

$$\frac{dV}{dt} = \frac{25\pi}{432}\cdot 3h^2\frac{dh}{dt}$$

$\frac{dV}{dt} = -10\frac{ft^3}{min} \Rightarrow$ $$-10\frac{ft^3}{min} = \frac{75\pi}{432}h^2\frac{dh}{dt}$$

$h = 8ft \Rightarrow$ $$-10\frac{ft^3}{min} = \frac{75\pi}{432}(8ft)^2\frac{dh}{dt}$$

$$-\frac{10ft^3}{min} = \frac{4800\pi ft^2}{432}\frac{dh}{dt}$$

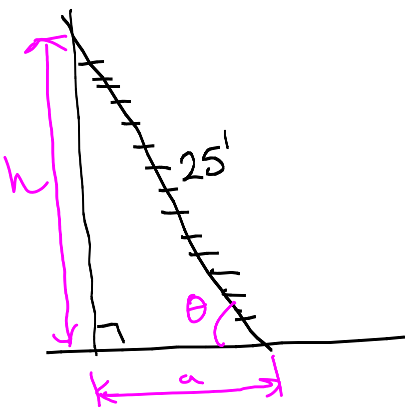
$$-\frac{10ft^3}{min}\cdot\frac{432}{4800\pi ft^2} = \frac{dh}{dt}$$

$$-\frac{432}{480\pi}\frac{ft}{min} = \frac{dh}{dt}$$

$$\boxed{-\frac{9}{10\pi}\frac{ft}{min} = \frac{dh}{dt}}$$

$$\approx \boxed{-0.286\ ft/min}$$

Example 3: A ladder 25 feet long is leaning against a wall. The base of the ladder is pulled away at 2 feet per second. How fast is the top of the ladder moving when the base is 9 feet away? What about when it is 24 feet away? How fast is the angle between the ladder and the floor changing when the base is 9 ft away?



Know: $\frac{da}{dt} = +2$ ft/sec

Want: $\frac{dh}{dt}$

When: $a = 9$ ft
$a = 24$ ft

Also: Want: $\frac{d\theta}{dt}$

When: $a = 9$ ft

$$a^2 + h^2 = 25^2$$

$$\frac{d}{dt}(a^2+h^2) = \frac{d}{dt}(625)$$

$$2a\frac{da}{dt} + 2h\frac{dh}{dt} = 0$$

$$2h\frac{dh}{dt} = -2a\frac{da}{dt}$$

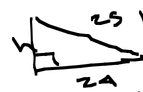
$$\frac{dh}{dt} = \frac{-2a\frac{da}{dt}}{2h} = -\frac{a}{h}\frac{da}{dt}$$

We know a, we know $\frac{da}{dt}$, must find h:

$a = 9$ ft

$h^2 + 9^2 = 25^2$
$h^2 = 625 - 81$
$= 544$
$h = \sqrt{544}$

$a = 24$ ft



$h^2 + 24^2 = 25^2$
$h^2 = 625 - 576$
$= 49$
$h = 7$

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Example 4: A particle is moving along the parabola $y^2 = 4x + 8$. As it passes through the point (7, 6) its y-coordinate is increasing at the rate of 3 units per second. How fast is the x-coordinate changing at this instant?

Ex 3 cont'd

$$\frac{dh}{dt} = \frac{-2a\frac{da}{dt}}{2h} = -\frac{a}{h}\frac{da}{dt}$$

$a = 9\text{ ft}$, $\frac{da}{dt} = 2\text{ ft/s}$, $h = \sqrt{544}\text{ ft}$ $\Rightarrow$ $\frac{dh}{dt} = -\frac{9\,\text{ft}}{\sqrt{544}\,\cancel{\text{ft}}} \cdot 2\,\cancel{\text{ft}}/\text{s}$

$$= -\frac{18}{\sqrt{544}}\text{ ft/s} \approx \boxed{-0.7717\text{ ft/s}}$$

$a = 24\text{ ft}$, $\frac{da}{dt} = 2\text{ ft/s}$, $h = 7\text{ ft}$ $\Rightarrow$ $\frac{dh}{dt} = -\frac{a}{h}\frac{da}{dt} = -\frac{24\text{ ft}}{7\text{ ft}} \cdot 2\text{ ft/s} = -\frac{48}{7}\frac{\text{ft}}{\text{sec}}$

$$= \boxed{-6\tfrac{6}{7}\text{ ft/sec}}$$

Now we need $\frac{d\theta}{dt}$ when $h = 9$ ft.

Want an equation relating θ and a to each other:

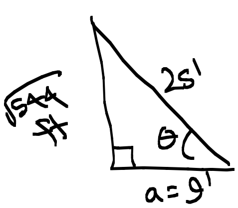
$$\cos\theta = \frac{a}{25}$$

$$\cos\theta = \frac{1}{25}\cdot a$$

$$\frac{d}{dt}(\cos\theta) = \frac{d}{dt}\left(\frac{1}{25}a\right)$$

$$-\sin\theta\frac{d\theta}{dt} = \frac{1}{25}\frac{da}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{25\sin\theta}\cdot\frac{da}{dt}$$ Need to find θ, or $\sin\theta$

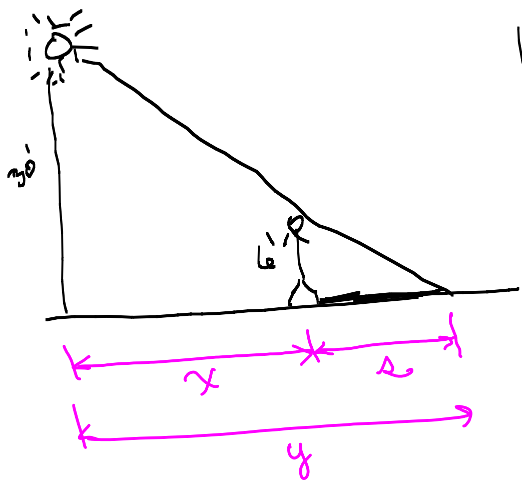


so $\sin\theta = \frac{\sqrt{544}}{25}$

So, $$\frac{d\theta}{dt} = -\frac{1}{25\sin\theta}\frac{da}{dt} = -\frac{1}{\cancel{25}\,\text{ft}\left(\frac{\sqrt{544}}{\cancel{25}}\right)}\cdot 2\text{ ft/sec}$$

$$= \boxed{-\frac{2}{\sqrt{544}}\ /\text{sec} \approx}$$

Example 5: A 6-foot tall man walks away from a 30-foot tall lamppost at a speed of 400 feet per minute. When he is 50 feet away from the lamppost, at what rate is his shadow lengthening? How fast is the tip of his shadow moving?



Know: $\frac{dx}{dt} = 400 \text{ ft/min}$

Want: $\frac{ds}{dt}$, $\frac{dy}{dt}$

When: $x = 50'$

$x + s = y$ must get rid of a variable

Similar triangles: $\frac{6}{s} = \frac{30}{y}$

$\frac{6}{s} = \frac{30}{x+s}$

$6(x+s) = 30s$

$6x + 6s = 30s$

$6x = 24s$

$\frac{d}{dt}(6x) = \frac{d}{dt}(24s)$

$6\frac{dx}{dt} = 24\frac{ds}{dt}$

$\frac{6}{24}\frac{dx}{dt} = \frac{ds}{dt}$

$\frac{ds}{dt} = \frac{1}{4}\frac{dx}{dt} = \frac{1}{4}(400 \text{ ft/min}) = 100 \text{ ft/min}$

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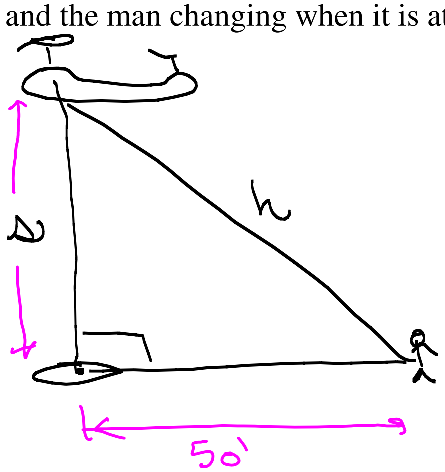
helipad
landing pad

Example 6: At a distance of 50 ft from the pad, a man observes a helicopter taking off from a heliport. The helicopter is rising vertically at a speed of 44 ft/second. How fast is the distance between the helicopter and the man changing when it is at an altitude of 120 ft?



Know: $\frac{ds}{dt} = +\frac{44 \text{ ft}}{\text{sec}}$

Want: $\frac{dh}{dt}$

When: $s = 120 \text{ ft}$

Pythagorean Thm:

$s^2 + (50 \text{ ft})^2 = h^2$

$\frac{d}{dt}(s^2 + 2500 \text{ ft}^2) = \frac{d}{dt}(h^2)$

$2s\frac{ds}{dt} + 0 = 2h\frac{dh}{dt}$

$\frac{2s}{2h} \cdot \frac{ds}{dt} = \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{s}{h} \cdot \frac{ds}{dt}$

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Ex 5 cont'd:

To find $\frac{dy}{dt}$, note that $x + s = y$.

$$\frac{d}{dt}(x+s) = \frac{d}{dt}(y)$$

$$\frac{dx}{dt} + \frac{ds}{dt} = \frac{dy}{dt}$$

Substitute $\frac{dx}{dt} = 400$ ft/min, $\frac{ds}{dt} = 100$ ft/min:

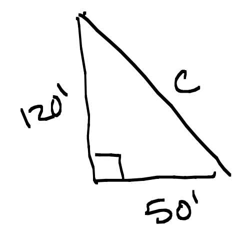
$$\frac{400 \text{ ft}}{\text{min}} + \frac{100 \text{ ft}}{\text{min}} = \frac{dy}{dt}$$

$$\frac{dy}{dt} = 500 \text{ ft/min}$$

$\boxed{\text{The shadow is lengthening at a rate of 100 ft/min, and the tip of the shadow is moving at 500 ft/min.}}$

Ex 6 cont'd:

$$\frac{dh}{dt} = \frac{s}{h} \cdot \frac{ds}{dt}$$



When $s = 120$ ft, what is h?

$$50^2 + 120^2 = c^2$$

$$2500 + 14400 = c^2$$

$$16900 = c^2$$

$$130' = c$$

$$\frac{dh}{dt} = \frac{s}{h} \cdot \frac{ds}{dt} = \frac{120 \text{ ft}}{130 \text{ ft}} \cdot 44 \text{ ft/sec} \approx \boxed{40.615 \text{ ft/sec}}$$

Homework Questions

2.5 #25 $(x+y)^3 = x^3+y^3$. Find $\frac{dy}{dx}$ at $(-1,1)$

$$\frac{d}{dx}(x+y)^3 = \frac{d}{dx}(x^3+y^3)$$

$$3(x+y)^2\frac{d}{dx}(x+y) = 3x^2 + 3y^2\frac{dy}{dx}$$

$$3(x+y)^2\left(1+\frac{dy}{dx}\right) = 3x^2+3y^2\frac{dy}{dx}$$

$$3(x+y)^2 + 3(x+y)^2\frac{dy}{dx} = 3x^2+3y^2\frac{dy}{dx}$$

$$3(x+y)^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 3x^2 - 3(x+y)^2$$

$$\frac{dy}{dx}\left(3(x+y)^2 - 3y^2\right) = 3x^2-3(x+y)^2$$

$$\frac{dy}{dx} = \frac{3x^2-3(x+y)^2}{3(x+y)^2-3y^2}$$

Now put in $x=-1$, $y=1$

2.5 #41b $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Show the eqn to tangent line at (x_0,y_0) is $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$

$$\frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1$$

$$\frac{d}{dx}\left(\frac{1}{a^2}x^2+\frac{1}{b^2}y^2\right) = \frac{d}{dx}(1)$$

$$\frac{1}{a^2}(2x) + \frac{1}{b^2}\left(2y\frac{dy}{dx}\right) = 0$$

$$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = -\frac{2x}{a^2}\cdot\frac{b^2}{2y} = -\frac{xb^2}{ya^2}$$

$$m\Big|_{\substack{x=x_0\\y=y_0}} = -\frac{x_0b^2}{y_0a^2}$$

Write eqn: $y-y_0 = m(x-x_0)$

$$y-y_0 = -\frac{x_0b^2}{y_0a^2}(x-x_0)$$

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$$y - y_0 = -\frac{x_0 b^2}{y_0 a^2}(x - x_0)$$

Multiply both sides by $y_0 a^2$:

$$y y_0 a^2 - y_0^2 a^2 = -x_0 x b^2 + x_0^2 b^2$$

$$y y_0 a^2 + x_0 x b^2 = y_0^2 a^2 + x_0^2 b^2$$

$$y y_0 a^2 - y_0^2 a^2 = x_0^2 b^2 - x_0 x b^2$$

$$\frac{a^2(y y_0 - y_0^2)}{a^2 b^2} = \frac{b^2(x_0^2 - x_0 x)}{a^2 b^2}$$

Note: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

so $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$

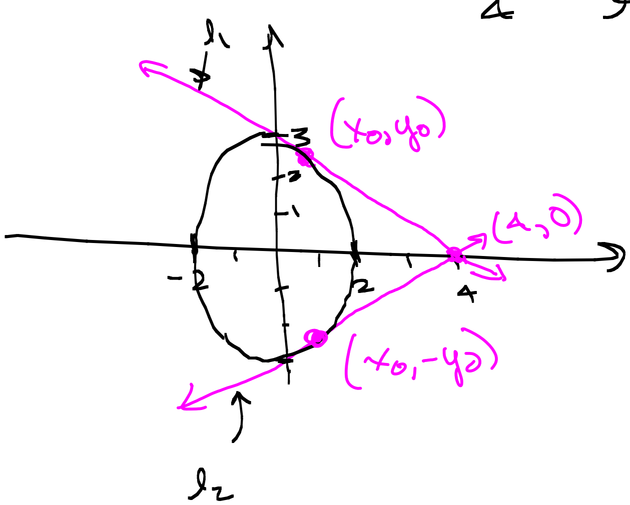
$$\frac{a^2 y y_0}{a^2 b^2} - \frac{a^2 y_0^2}{a^2 b^2} = \frac{b^2 x_0^2}{a^2 b^2} - \frac{x_0 x b^2}{a^2 b^2}$$

$$\frac{y y_0}{b^2} - \frac{y_0^2}{b^2} = \frac{x_0^2}{a^2} - \frac{x_0 x}{a^2}$$

$$\frac{y y_0}{b^2} = \underbrace{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}}_{=1} - \frac{x_0 x}{a^2}$$

$$\frac{y y_0}{b^2} = 1 - \frac{x_0 x}{a^2} \checkmark$$

73 Find the eqns of both tangent lines to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ that pass through $(4,0)$



$$\frac{dy}{dx} = -\frac{9x}{4y}$$

Slope of l_1: (from derivative) $m = -\frac{9x_0}{4y_0}$

Slope of l_1 (from algebra) $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{y_0 - 0}{x_0 - 4}$$

$$= \frac{y_0}{x_0 - 4}$$

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m from derivative = m from algebra:

$$-\frac{9x_0}{4y_0} = \frac{y_0}{x_0-4}$$

Do same thing for l_2:

derivative, $m = \frac{-9x_0}{4(-y_0)} = \frac{9x_0}{4y_0}$ (positive)

algebra: $\frac{y_2-y_1}{x_2-x_1} = \frac{-y_0-0}{x_0-4} = \frac{-y_0}{x_0-4}$

Set equal: $\frac{-y_0}{x_0-4} = \frac{9x_0}{4y_0}$

From eqn of ellipse: $\frac{x_0^2}{4} + \frac{y_0^2}{9} = 1$

$$\frac{x_0^2}{4} = 1 - \frac{y_0^2}{9}$$

$$x_0^2 = 4 - \frac{4y_0^2}{9}$$

$$x_0 = \pm\sqrt{4 - \frac{4y_0}{9}}$$

$$\frac{y_0^2}{9} = 1 - \frac{x_0^2}{4}$$

$$y_0^2 = 9 - \frac{9x_0^2}{4}$$

Choose $x_0 \geq 0$ because it's in Quad I

$$-\frac{9x_0}{4y_0} = \frac{y_0}{x_0-4} \Rightarrow -9x_0(x_0-4) = 4y_0^2$$

Put in $y_0^2 = 9 - \frac{9x_0^2}{4}$:

$$-9x_0(x_0-4) = 4\left(9 - \frac{9x_0^2}{4}\right)$$

$$-9x_0(x_0-4) = 36 - 9x_0^2$$

$$\cancel{-9x_0^2} + 36x_0 = 36 \cancel{-9x_0^2}$$

$$36x_0 = 36$$

$$x_0 = 1$$

Yea! Now find y_0

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#73 cont'd

$$y_0^2 = 9 - \frac{9x_0^2}{4}$$

$$x_0 = 1 \Rightarrow y_0^2 = 9 - \frac{9}{4} = \frac{36}{4} - \frac{9}{4} = \frac{27}{4}$$

$$y_0 = \pm\sqrt{\frac{27}{4}} = \pm\frac{\sqrt{27}}{2} = \pm\frac{3\sqrt{3}}{2}$$

we put (x_0, y_0) in Quadrant I, so $y_0 = +\frac{3\sqrt{3}}{2}$

For l_1: $\quad m = \dfrac{-9x_0}{4y_0} = \dfrac{-9(1)}{4\left(\frac{3\sqrt{3}}{2}\right)} = \dfrac{-9}{6\sqrt{3}} = -\dfrac{9\sqrt{3}}{18} = -\dfrac{\sqrt{3}}{2}$

Finally: $\quad y - y_1 = m(x - x_1)$

Using $(x_1, y_1) = (4, 0)$: $\quad y - 0 = -\frac{\sqrt{3}}{2}(x - 4)$

$$y = -\frac{\sqrt{3}}{2}x + \frac{4\sqrt{3}}{2}$$

$$y = -\frac{\sqrt{3}}{2}x + 2\sqrt{3}$$

— Then find l_2 (you'll need to use the slope for l_2 we calculated earlier)