3.1: Extrema on an Interval

Absolute maximum and minimum: or global maximum

If $f(x) \le f(c)$ for every x in the domain of *f*, then $f(c)$ is the *maximum*, or *absolute maximum*, of *f*.

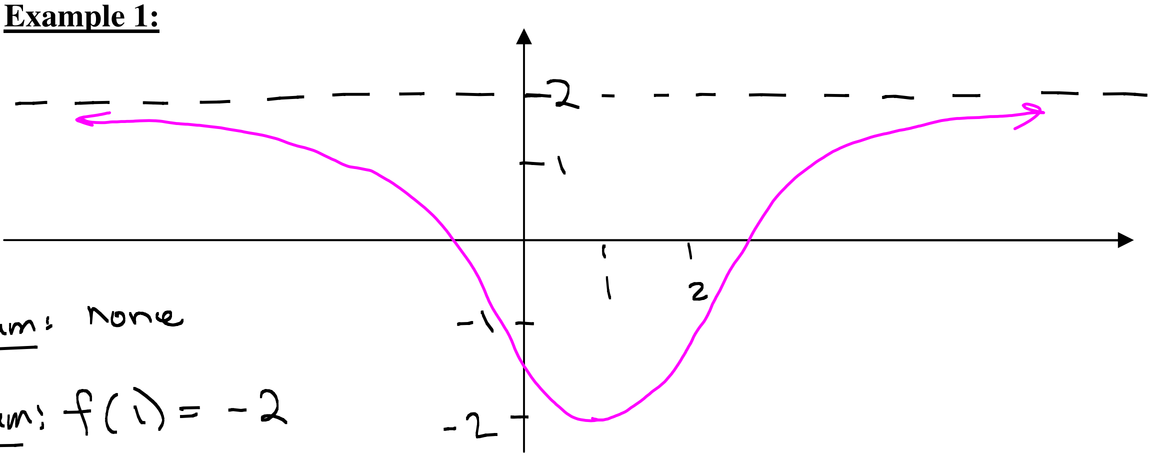
If $f(x) \ge f(c)$ for every x in the domain of *f*, then $f(c)$ is the *minimum*, or *absolute minimum* of *f*.

The maximum and minimum values of a function are called the *extreme values* (extrema) of the function.

In other words,

- The *absolute maximum* is the largest *y*-value on the graph.
- The *absolute minimum* is the smallest *y*-value on the graph.

Example 1:



Maximum: none

Minimum: $f(1) = -2$

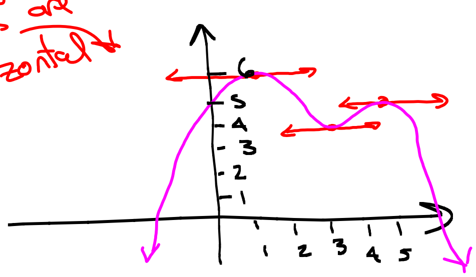
Maxima: plural of maximum

Minima: plural of minimum

Relative (Local) Maxima and Minima:

- A function *f* has a *relative maximum*, or *local maximum*, at $x = c$ if there is an interval (a,b) around *c* such that $f(x) \le f(c)$ for every *x* in (a,b). (These are the "hilltops").
- A function *f* has a *relative minimum*, or *local minimum*, at $x = c$ if there is an interval (a,b) around *c* such that $f(x) \ge f(c)$ for every *x* in (a,b). (These are the "bottoms of valleys").

tangent lines are horizontal

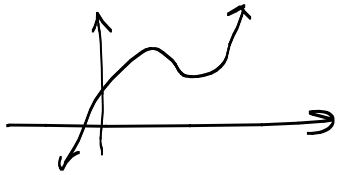


Relative maxima: $f(1) = 6$ and $f(5) = 5$

Absolute maximum: $f(1) = 6$

Relative minima: $f(3) = 4$

No absolute minimum.



No absolute min, no absolute max.
It has one relative min and one relative max

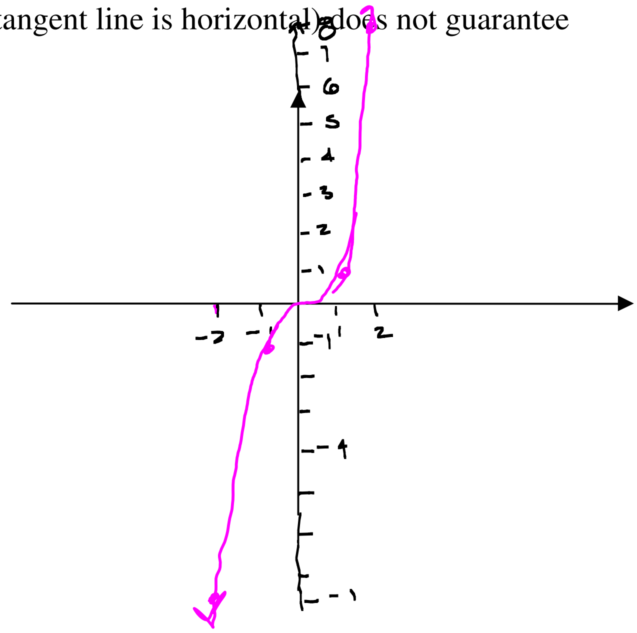
Notice: If the function is differentiable (smooth), then the tangent line at a local minimum or maximum is horizontal.

> Fermat's Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c)=0$.

This means that if f is differentiable at c and has a relative extreme at c, then the tangent line to f at c must be horizontal.

However, we must be careful. The fact that $f'(c)=0$ (tangent line is horizontal) does not guarantee that there is a relative minimum or maximum at c.

Example 2: $f(x)=x^3$



$f'(x)=3x^2$

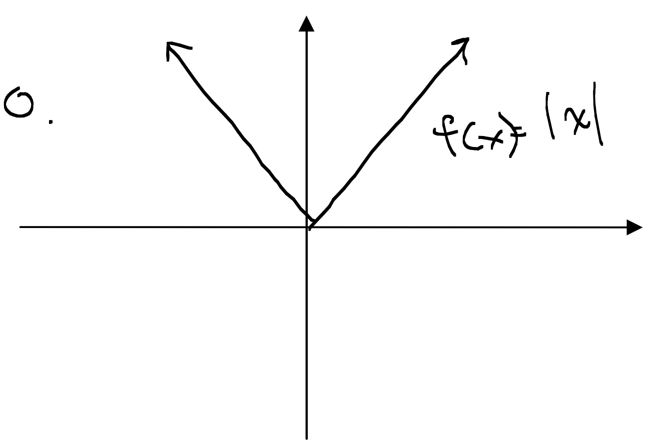
$f'(0)=3(0)^2=0$

So tangent line at $(0,0)$ is horizontal.

No absolute max or min.
No relative max or min

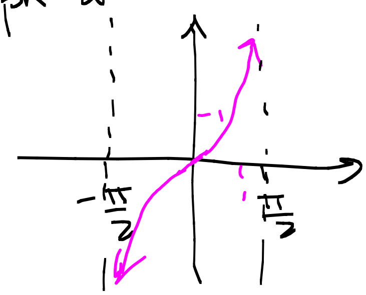
Example 3: There can be a local maximum or minimum at c even if $f'(c)$ does not exist.

Relative (and absolute) minimum is $f(0)=0$.



But $f'(0)$ does not exist (sharp corner at $(0,0)$)

Note: graph of $f(x)=\tan x$



$f'(x)=\sec^2 x$

$f'(0)=\sec^2(0)=\left(\frac{1}{\cos(0)}\right)^2=\left(\frac{1}{1}\right)^2=1$

Slope of tangent line at $(0,0)$ is 1

Critical numbers:

Critical Number: A *critical number* of a function *f* is a number *c* in the domain of *f* such that either $f'(c)=0$ or $f'(c)$ does not exist.

Theorem: If *f* has a local maximum or minimum at *c*, then *c* is a critical number of *f*.

Note: The converse of this theorem is not true. It is possible for *f* to have a critical number at *c*, but not to have a local maximum or minimum at *c*.

Example 4: Find the critical numbers of $f(x)=x^3+\frac{17}{2}x^2-6x+4$.

Find f' and set it equal to 0.

$$f'(x) = 3x^2 + \frac{17}{2}(2x) - 6$$
$$= 3x^2 + 17x - 6$$

set $f'(x)=0$

$0 = 3x^2+17x-6$

$0=(3x-1)(x+6)$

$3x-1=0$, $3x=1$, $x=\frac{1}{3}$ | $x+6=0$, $x=-6$

$\boxed{\text{Critical numbers are } \frac{1}{3}, -6.}$

Example 5: Find the critical numbers of $f(x)=x^{2/3}$

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

Note: Setting $\frac{2}{3\sqrt[3]{x}}=0$ multiply by $3\sqrt[3]{x}$ gives us $2=0$, which has no solution.

$f'(x)$ is undefined for $x=0$

$f'(x)$ is never 0. (So $f'(0)$ does not exist)

Is 0 in the domain of f?

$f(x) = x^{2/3} = \sqrt[3]{x^2}$

Domain of f: $(-\infty,\infty)$

$\boxed{\text{Critical Number: } 0}$

Example 6: Find the critical numbers of $f(x)=\dfrac{x^2}{x-3}$

Domain: $x\neq 3$

$(-\infty,3)\cup(3,\infty)$

$$f'(x) = \frac{(x-3)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(x-3)}{(x-3)^2}$$
$$= \frac{(x-3)(2x) - x^2(1)}{(x-3)^2}$$
$$= \frac{2x^2-6x-x^2}{(x-3)^2}$$
$$= \frac{x^2-6x}{(x-3)^2} = \frac{x(x-6)}{(x-3)^2}$$

$$f'(x) = \frac{x(x-6)}{(x-3)^2}$$

$f'(x)$ is undefined for $x=3$.

$f'(x)=0$ for $x=0$, $x=6$.

Note: $0 = \frac{x(x-6)}{(x-3)^2}$

$0 = x(x-6)$

$x=0$, $x=6$

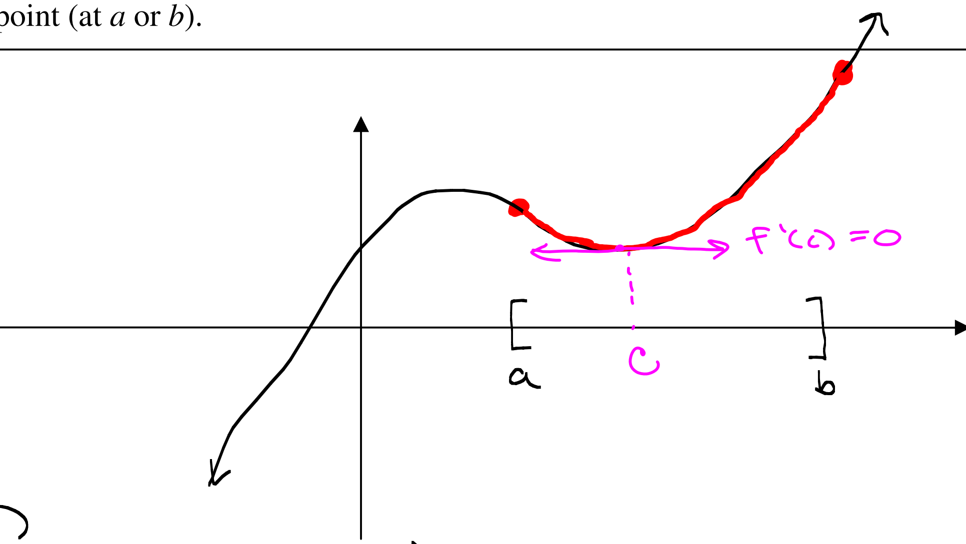
$\boxed{\text{Critical Numbers: } 0, 6}$

(3 is not in the domain, so can't be a critical number

Absolute extrema on a closed interval:

> Extreme Value Theorem: If *f* is continuous on a closed interval $[a,b]$, then *f* has both an absolute maximum and an absolute minimum on $[a,b]$.
>
> Note: The absolute maximum and the absolute minimum must occur at either a critical value in (a,b) or at an endpoint (at *a* or *b*).

Example 7:

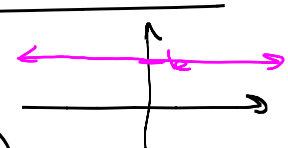


Absolute max: f(b)
(at endpoint)

Absolute min: f(c)
(at a critical number in the interior)

Does a horizontal line have an absolute minimum or maximum?

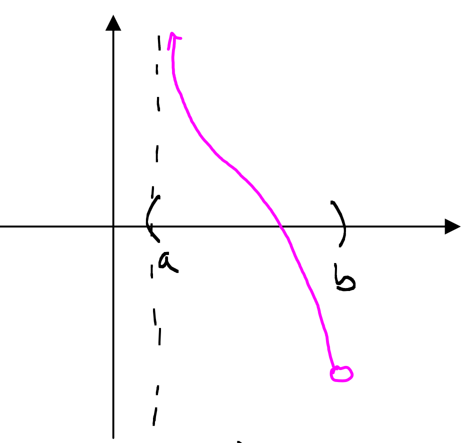
f(x) = k
Absolute min: k
Absolute max: k



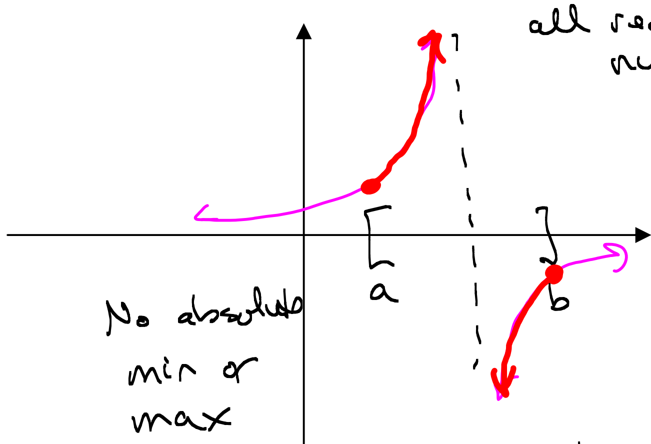
both occur at all real numbers

Example 8: If either hypothesis of the extreme value theorem (continuity or closed interval) is violated, the existence of an absolute maximum or minimum is not guaranteed.

No absolute min or max.



Continuous on (a, b) but not on [a, b]



No absolute min or max

Discontinuity in the middle of [a, b]

Step 0. Verify that f is continuous on $[a, b]$.

Process for Finding the Absolute Extrema of a Continuous Function on a Closed Interval:

1. Find the critical values in (a,b).
2. Compute the value of f at each critical value in (a,b) and also compute $f(a)$ and $f(b)$.
3. The absolute maximum is the largest of these y-values and the absolute minimum is the smallest of these y-values.

Example 9: Find the absolute extrema for $f(x) = x^2 + 2$ on the interval $[-2,3]$.

Is it continuous? Yes

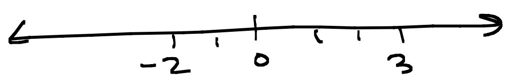
Find critical numbers: $f'(x) = 2x$

Set $f'(x)=0$: $2x=0$

$x = \frac{0}{2} = 0$

Critical number: 0

Note $0 \in [-2,3]$



x	$f(x)$
0	$f(0)=0^2+2 = 2$
-2	$f(-2)=(-2)^2+2 = 6$
3	$f(3)=3^2+2 = 11$

smallest

largest

$\boxed{\begin{array}{l}\text{Absolute max: } f(3)=11 \\ \text{Absolute min: } f(0)=2\end{array}}$

Example 10: Find the extreme values of $g(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 - 2x^2 + 3$ on the interval $[-2,1]$.

g is continuous on $(-\infty, \infty)$.

$g'(x) = \frac{1}{2}(4x^3) - \frac{2}{3}(3x^2) - 4x$

$= 2x^3 - 2x^2 - 4x$

$= 2x(x^2 - x - 2)$

$= 2x(x-2)(x+1)$

Critical numbers: $0, 2, -1$

Only 0 and -1 are in interval $[-2,1]$.

Find y-values:

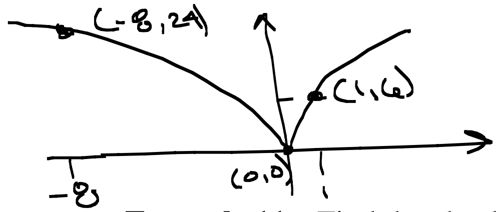
$g(0) = \frac{1}{2}(0)^4 - \frac{2}{3}(0)^3 - 2(0)^2 + 3 = 3$

$g(-1) = \frac{1}{2}(-1)^4 - \frac{2}{3}(-1)^3 - 2(-1)^2 + 3 = \frac{13}{6} = 2\frac{1}{6}$

$g(-2) = \frac{1}{2}(-2)^4 - \frac{2}{3}(-2)^3 - 2(-2)^2 + 3 = \frac{25}{3} = 8\frac{1}{3}$

$g(1) = \frac{1}{2} - \frac{2}{3} - 2 + 3 = \frac{5}{6}$

$\boxed{\begin{array}{l}\text{Absolute maximum is } g(-2) = \frac{25}{3}. \\ \text{Absolute minimum is } g(1) = \frac{5}{6}.\end{array}}$



Example 11: Find the absolute extrema of $h(x) = 6x^{2/3}$ on the intervals (a) $[-8,1]$, (b) $[-8,1)$, and. (c) $(-8,1)$.

$h(x) = 6x^{2/3} = 6\sqrt[3]{x^2}$. Domain is $(-\infty,\infty)$.

h is continuous on its domain.

$h'(x) = 6\left(\frac{2}{3}x^{-\frac{1}{3}}\right) = 4x^{-1/3} = \frac{4}{\sqrt[3]{x}}$

h' is undefined for $x=0$.
0 is in domain of h, so 0 is a critical number.

h' is never 0 (because numerator is never 0).

0 is the only critical number

Find y-values.

$h(0) = 6\sqrt[3]{(0)^2} = 0$

$h(-8) = 6\sqrt[3]{(-8)^2} = 6\sqrt[3]{64} = 6 \cdot 4 = 24$

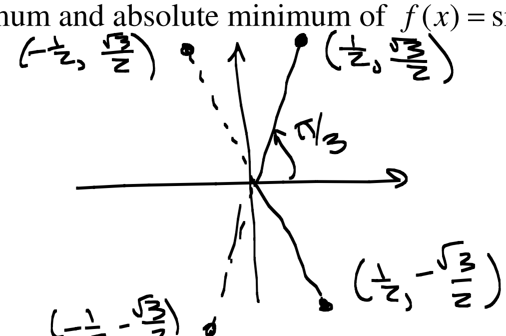
$h(1) = 6\sqrt[3]{(1)^2} = 6 \cdot 1 = 6$

(a) On $[-8,-1]$, absolute minimum is $h(0)=0$, and absolute maximum is $h(-8)=24$.

(b) on $[-8,1)$, absolute max is still $h(-8)=24$ and abs. min is $h(0)=0$.

(c) on $(-8,1)$, the abs. min is $h(0)=0$. No absolute max.

Example 12: Find the absolute maximum and absolute minimum of $f(x) = \sin 2x - x$ on the interval $[0,\pi]$.



f is continuous on $(-\infty,\infty)$.

$f'(x) = \cos(2x)(2) - 1$
$= 2\cos 2x - 1$

Set $f'(x)=0$: $0 = 2\cos(2x) - 1$
$1 = 2\cos(2x)$
$\frac{1}{2} = \cos(2x)$
$2x = \frac{\pi}{3}$, $2x = \frac{5\pi}{3}$
$x = \frac{\pi}{6}$ $x = \frac{5\pi}{6}$ both in $[0,\pi]$

$0 \le x \le \pi$ Given
$0 \le 2x \le 2\pi$

$f(\frac{\pi}{6}) = \sin(\frac{2\pi}{6}) - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \approx 0.34247$

$f(\frac{5\pi}{6}) = \sin(\frac{10\pi}{6}) - \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{5\pi}{6} \approx -3.48402$

$f(0) = \sin(2\cdot 0) - 0 = \sin 0 - 0 = 0$

$f(\pi) = \sin(2\pi) - \pi = 0 - \pi = -\pi \approx -3.14159$

Absolute maximum is $f(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$.
Absolute minimum is $f(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2} - \frac{5\pi}{6}$.