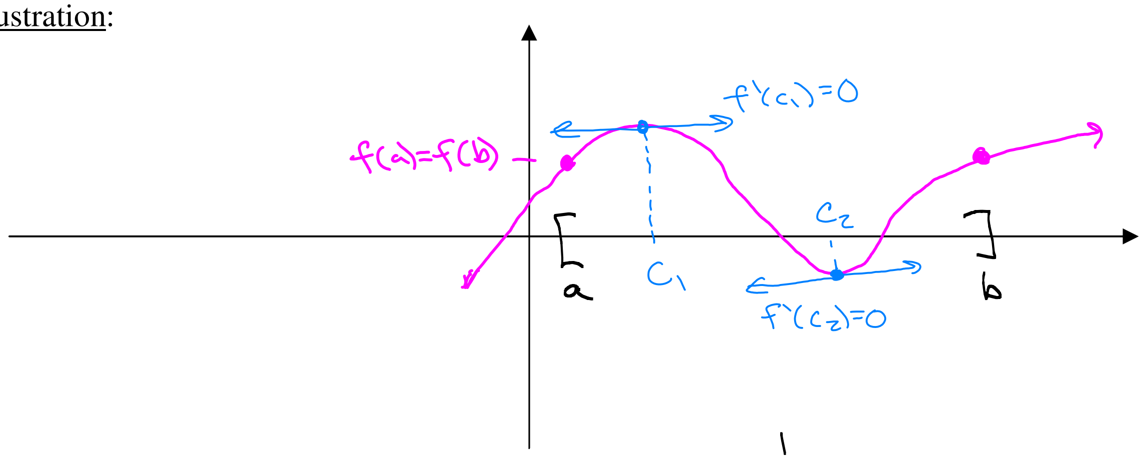
3.2: Rolle's Theorem and the Mean Value Theorem

Rolle's Theorem:

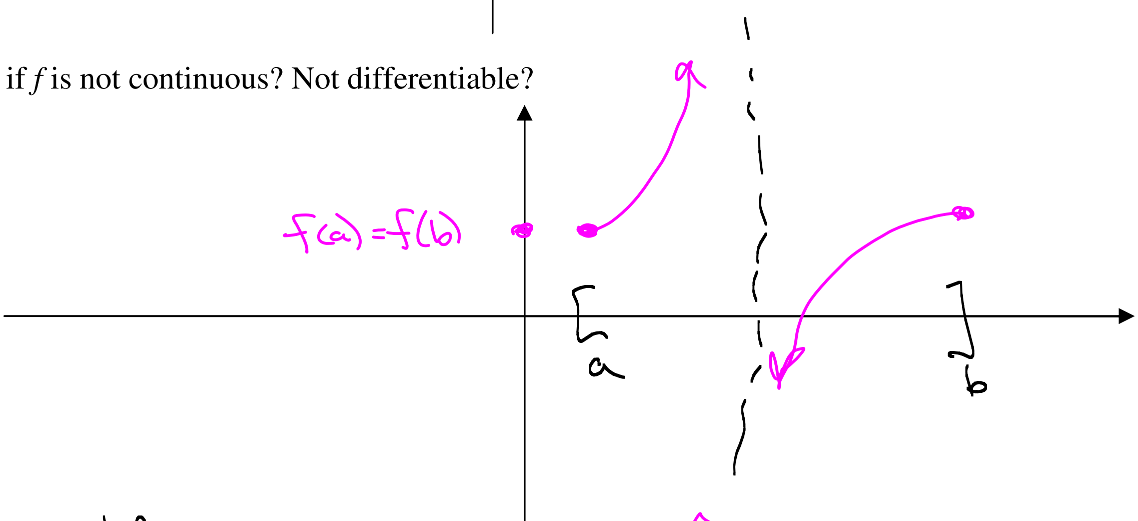
Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

If $f(a) = f(b)$, then there is a number c in (a,b) such that $f'(c) = 0$.

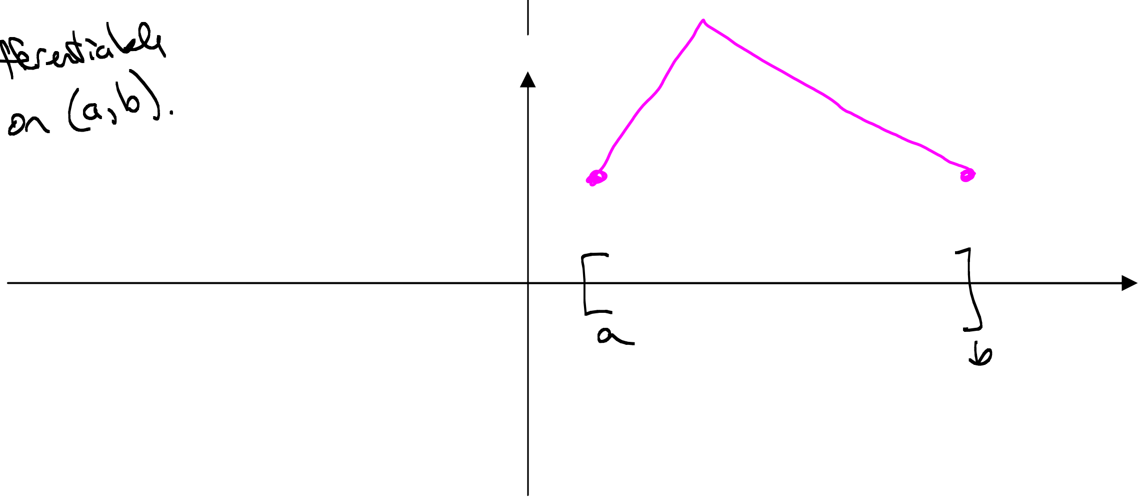
Illustration:



What if f is not continuous? Not differentiable?



Not differentiable on (a,b).



Example 1: Show that the function $f(x) = x^2 - 4x - 5$ satisfies the hypotheses of Rolle's Theorem on the interval $[-1,5]$. Find all numbers c in $[-1,5]$ that satisfy the conclusion of Rolle's Theorem.

f is continuous and differentiable on $(-\infty,\infty)$.

Show that $f(-1)=f(5)$: $f(-1) = (-1)^2 - 4(-1) - 5$

$= 1 + 4 - 5 = 0$

$f(5) = 5^2 - 4(5) - 5$

$= 25 - 25 = 0$

So $f(-1) = f(5)$.

$f'(x) = 2x - 4$

$= 2(x-2)$

$f'(c) = 0 \Rightarrow 0 = 2(x-2)$ This is a

$x = 2$. Critical number.

$\boxed{\text{2 is the only } c \text{ in } [-1,5] \text{ such that } f'(c)=0.}$

Example 2: Show that the function $g(x) = -2x^4 + 16x^2$ satisfies the hypotheses of Rolle's Theorem on the interval $[-3,3]$. Find all numbers c that satisfy the conclusion of Rolle's Theorem.

g is continuous and differentiable on $(-\infty,\infty)$.

$g'(x) = -8x^3 + 32x$

$= -8x(x^2 - 4)$

$= -8x(x+2)(x-2)$

0, −2, 2 are the critical numbers. All are in $[-3,3]$.

Verify that $g(-3) = g(3)$:

$$\left.\begin{aligned} g(-3) &= -2(-3)^4 + 16(-3)^2 \\ g(3) &= -2(3)^4 + 16(3)^2 \end{aligned}\right\}$$ must be equal! No need to calculate exactly.

$\boxed{\text{0, −2, 2 are the c-values that satisfy the theorem.}}$

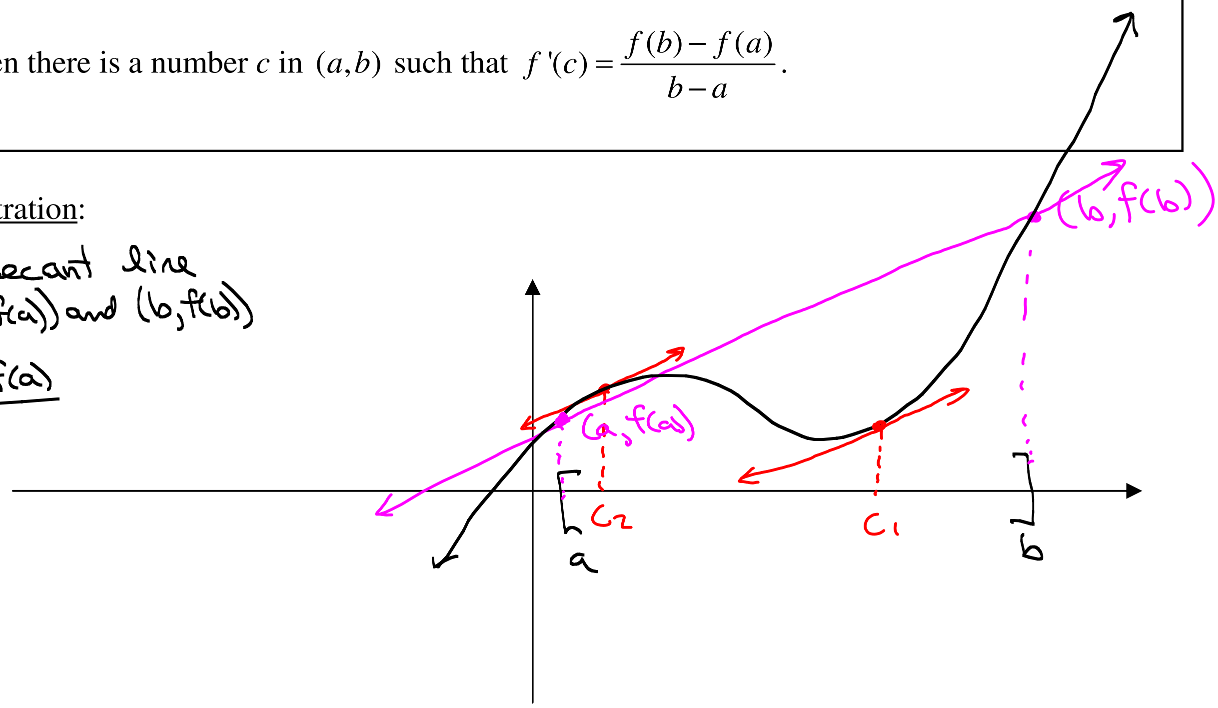
<u>Mean Value Theorem</u>:

Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).

Then there is a number c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

<u>Illustration</u>:

Slope of secant line joining $(a,f(a))$ and $(b,f(b))$ is $\dfrac{f(b)-f(a)}{b-a}$



A few consequences of the Mean Value Theorem:

1) The Mean Value Theorem guarantees the existence of a tangent line parallel to the secant line that contains the endpoints $(a, f(a))$ and $(b, f(b))$.

2) In terms of rates of change, the Mean Value Theorem guarantees that there is some point at which the <u>instantaneous</u> rate of change is equal to the <u>average</u> rate of change over $[a,b]$.

3) If the derivative of a function is 0 for every number in an interval, then the function is constant on that interval.

4) If two functions have the same derivative on an interval, they differ by a constant on that interval.

Example 3: Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$f(x) = \sqrt{x} - 2x$ on the interval $[0, 4]$

Want to find the c's where the slopes of the tangent line is equal to the slope of the secant line connecting the endpoints.

Domain of f: $x \geq 0$

f is continuous on $[0, \infty)$

$f(x) = x^{1/2} - 2x$

$f'(x) = \frac{1}{2}x^{-1/2} - 2$

$= \frac{1}{2\sqrt{x}} - 2$

$f'(x)$ is undefined only at 0.

f is differentiable on $(0, \infty)$.

So hypotheses of the Mean Value Theorem are satisfied.

Find slope of secant line:

$f(0) = \sqrt{0} - 2(0) = 0$ ordered pair $(0,0)$

$f(4) = \sqrt{4} - 2(4) = 2 - 8 = -6$. ordered pair $(4, -6)$.

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(4) - f(0)}{4 - 0} = \frac{-6 - 0}{4} = -\frac{3}{2}$$

Set $f'(x) = -\frac{3}{2}$:

$$\frac{1}{2\sqrt{x}} - 2 = -\frac{3}{2}$$

multiply by $2\sqrt{x}$:

$$\cancel{(2\sqrt{x})}\frac{1}{\cancel{2\sqrt{x}}} - 2(2\sqrt{x}) = -\frac{3}{\cancel{2}}(\cancel{2}\sqrt{x})$$

$$1 - 4\sqrt{x} = -3\sqrt{x}$$

$$1 = \sqrt{x}$$

$$(1)^2 = (\sqrt{x})^2$$

$$1 = x$$

So $c = 1$ is the only value that satisfies the conclusion of the mean Value Theorem.

Example 4: As you drive by a Houston police car, your speed is clocked at 50 miles per hour. Five minutes and five miles further, you pass by another HPD officer, who clocks you at 55 miles per hour. If the speed limit on the entire stretch of highway is 55 mph, would the second officer be justified in writing you a speeding ticket?

Ex 4:

Slope of secant line = average rate of change
= average velocity

slope of tangent line = instantaneous rate of change
= instantaneous velocity

$$\text{Average velocity} = \frac{\Delta \text{distance}}{\Delta \text{time}} = \frac{5 \text{ miles}}{5 \text{ minutes}}$$

$$= \frac{1 \text{ mile}}{\cancel{\text{min}}}\left(\frac{60 \cancel{\text{min}}}{1 \text{ hour}}\right) = 60 \text{ miles/hour}$$

Because of Mean Value Theorem, there must be a point c between the officers at which ~~my~~ the instantaneous velocity is equal to the average velocity of 60 mph.

$\boxed{\text{So yes, the ticket is justified!}}$:)