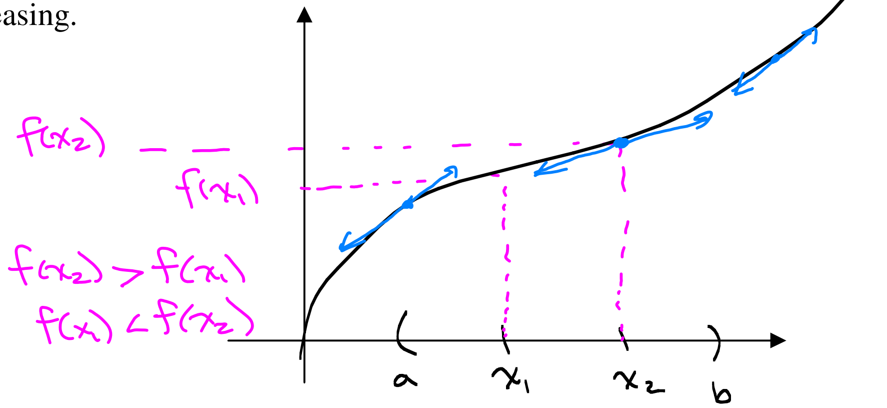
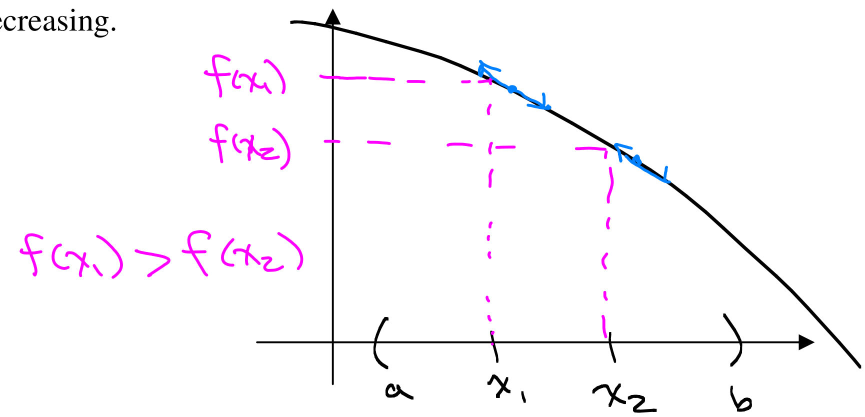
3.3: Increasing and Decreasing Functions and the First Derivative Test

Increasing and decreasing functions:

A function *f* is said to be *increasing* on the interval (a,b) if, for any two numbers x_1 and x_2 in (a,b), $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. A function *f* is *increasing at c* if there is an interval around *c* on which *f* is increasing.



A function *f* is said to be *decreasing* on the interval (a,b) if, for any two numbers x_1 and x_2 in (a,b), $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. A function *f* is *decreasing at c* if there is an interval around *c* on which *f* is decreasing.



Notice that wherever a function is increasing, the tangent lines have positive slope.
Notice that wherever a function is decreasing, the tangent lines have negative slope.

This means that we can use the derivative to determine the intervals where a function is increasing and decreasing.

> Increasing/Decreasing Test: Let f be a function that is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).
>
> - If $f'(x) > 0$ for every *x* in (a,b), then *f* is increasing on (a,b).
> - If $f'(x) < 0$ for every *x* in (a,b), then *f* is decreasing on (a,b).
> - If $f'(x) = 0$ for every *x* in (a,b), then *f* is constant on (a,b).

Example 1: $f(x) = x^2$

$f'(x) = 2x$

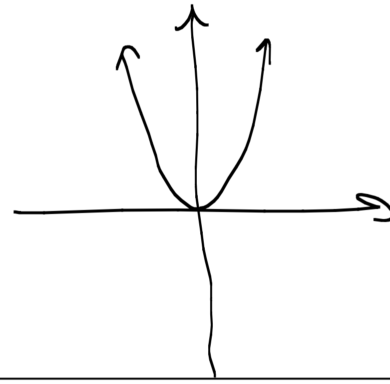
Derivative is 0 for $x = 0$. (only critical number).

$f'(x) > 0$ for $x > 0$.

$f'(x) < 0$ for $x < 0$.

f is increasing on $(0, \infty)$.

f is decreasing on $(-\infty, 0)$



Steps for Determining Increasing/Decreasing Intervals

1. Find all the values of x where $f'(x) = 0$ or where $f'(x)$ is not defined. Use these values to split the number line into intervals.

2. Choose a test number c in each interval and determine the sign of $f'(c)$.
   - If $f'(c) > 0$, then f is increasing on that interval.
   - If $f'(c) < 0$, then f is decreasing on that interval.

Note: Three types of numbers can appear on your number line:
1) Numbers where the function is defined and the derivative is 0. (These are critical numbers.)
2) Numbers where the function is defined and the derivative is undefined. (These are also critical numbers.)
3) Numbers where the function is undefined. (These are NOT critical numbers.)

First derivative test:

This procedure determines the relative extrema of a function *f*.

First derivative test:

Suppose that c is a critical number of a function f that is continuous on an open interval containing c.

- If $f'(x)$ changes from positive to negative across c, then f has a relative maximum at c.
- If $f'(x)$ changes from negative to positive across c, then f has a relative minimum at c.
- If $f'(x)$ does not change sign across c, then f does not have a relative extreme at c.

Example 2: Determine the intervals on which $f(x)=x^3+6x^2-36x+18$ is increasing and decreasing. Find the relative extrema.

$$f'(x) = 3x^2+12x-36$$
$$= 3(x^2+4x-12)$$
$$= 3(x+6)(x-2)$$

Set $f'(x)=0$: $0 = 3(x+6)(x-2)$

$x+6=0 \mid x-2=0$

Critical numbers: $-6, 2$

relative max

relative min

f': increasing $+ + + +$ | -6 | decreasing $- - - - - -$ | 2 | increasing $+ + + + + + + + +$

$(-\infty,-6)$: Choose test number $x=-7$.

$$f'(-7) = 3(-7)^2+12(-7)-36$$
$$= 3(49)-84-36$$
$$= 147 - 120$$
$$= 27 \text{ Positive}$$

OR

$$f'(-7) = 3(-7+6)(-7-2)$$
$$= 3(-1)(-9)$$
$$= 27 > 0$$

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Example 3: Determine the intervals on which $g(x)=x^3-6x^2+12x-8$ is increasing and decreasing. Find the relative extrema.

$$g'(x) = 3x^2-12x+12$$
$$= 3(x^2-4x+4)$$
$$= 3(x-2)^2$$

Only critical number is 2.

g: increasing $+ + + + + +$ | 2 | increasing $+ + + +$

$(-\infty,2)$: Test number $x=0$.

$$g'(0) = 3(0-2)^2$$
$$= 3(4)$$

$(+)$ increasing on $(-\infty,2)$

$(2,\infty)$: Test number $x=3$.

$$g'(3) = 3(3-2)^2 = 3(1)^2$$

$(+)$ increasing on $(2,\infty)$

$\boxed{g \text{ is increasing on } (-\infty,\infty).}$

$\boxed{\text{No relative extrema}}$

Ex 2 cont'd: $f'(x) = 3(x+6)(x-2)$

$(-6, 2)$: Test number $x = 0$.

$$f'(0) = 3(0+6)(0-2)$$
$$\Rightarrow (+)(+)(-)$$
$$\Rightarrow (-) \quad \text{decreasing on } (-6,2).$$

$(2,\infty)$: Test number $x = 3$.

$$f'(3) = 3(3+3)(3-2)$$
$$= 3(6)(1)$$
$$(+) \quad \text{increasing on } (2,\infty)$$

f is increasing on $(-\infty, -6)$ and on $(2,\infty)$.
f is decreasing on $(-6,2)$.

Relative max at $x = -6$.
Relative min at $x = 2$.
Find the y-values: $f(x) = x^3 + 6x^2 - 36x + 18$

$$f(-6) = (-6)^3 + 6(-6)^2 - 36(-6) + 18$$
$$= -\cancel{216} + \cancel{216} + 216 + 18$$
$$= 234$$

$$f(2) = 2^3 + 6(2)^2 - 36(2) + 18$$
$$= 8 + 24 - 72 + 18$$
$$= 32 - 72 + 18 = -40 + 18 = -22$$

Relative max: $f(-6) = 234$
Relative min: $f(2) = -22$

$$g(x) = x^{2/5} = \sqrt[5]{x^2}$$

Example 4: Determine the intervals on which $g(x) = x^{2/5}$ is increasing and decreasing. Find the relative extrema.

$$g'(x) = \frac{2}{5}x^{-3/5} = \frac{2}{5\sqrt[5]{x^3}}$$

Original function is defined on $(-\infty, \infty)$.

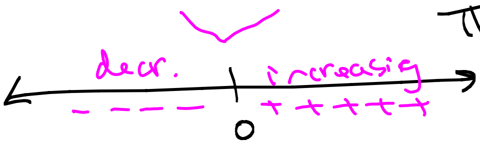
Derivative $g'(x)$ is undefined for $x = 0$.

So 0 is a critical number.

Where is $g'(x) = 0$? Nowhere.

The numerator is never 0.

g is increasing on $(0, \infty)$.
g is decreasing on $(-\infty, 0)$.
Relative min is $g(0) = 0$



$(0, \infty)$: Test $x = 1$:

$$g'(1) = \frac{2}{5\sqrt[5]{(1)^3}} = \frac{2}{5\sqrt[5]{1}} = \frac{2}{5(1)} = \frac{2}{5}$$

> 0 increasing

$(-\infty, 0)$: Test number $x = -1$.

$$g'(-1) = \frac{2}{5\sqrt[5]{(-1)^3}} = \frac{2}{5\sqrt[5]{-1}} = \frac{2}{5(-1)} = -\frac{2}{5} < 0$$

decr

Example 5: Determine the intervals on which $f(x) = x + \dfrac{4}{x}$ is increasing and decreasing. Find the relative extrema.

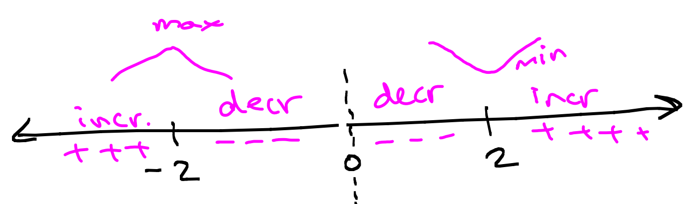
$$f(x) = \frac{x^2+4}{x}$$

f is undefined for $x = 0$.

$$f(x) = x + 4x^{-1}$$

$$f'(x) = 1 - 4x^{-2}$$

$$= 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2} = \frac{(x+2)(x-2)}{x^2}$$



$f'(x)$ is undefined for $x = 0$

$f'(x) = 0$ for $x = 2$, $x = -2$

$(-\infty, -2)$: Test $x = -3$

$$f'(-3) = \frac{(-3+2)(-3-2)}{(-3)^2} \Rightarrow \frac{(-)(-)}{(+)} \Rightarrow \frac{(+)}{(+)} \Rightarrow (+)$$

$(-2, 0)$: Test $x = -1$

$$f'(-1) = \frac{(-1+2)(-1-2)}{(-1)^2} \Rightarrow \frac{(+)(-)}{(+)} \Rightarrow (-)$$

$(0, 2)$: Test $x = 1$

$$f'(1) = \frac{(1+2)(1-2)}{1^2} \Rightarrow \frac{(+)(-)}{(+)} \Rightarrow (-)$$

$(2, \infty)$: Test $x = 3$

$$f'(3) = \frac{(3+2)(3-2)}{3^2} \Rightarrow \frac{(+)(+)}{(+)} \Rightarrow (+)$$

Increasing on $(-\infty, -2)$ and on $(2, \infty)$.

Decreasing on $(-2, 0)$ and also on $(0, 2)$.

Relative min is $f(2) = 4$

Relative max is $f(-2) = -4$

Find y-values for relative extrema: $f(2) = \dfrac{2^2+4}{2} = \dfrac{8}{2} = 4$

$$f(-2) = \frac{(-2)^2+4}{-2} = \frac{8}{-2} = -4$$

Example 6: Find the local extremes of $g(x)=(x^2-4)^{2/3}$. Where is it increasing and decreasing?

Example 7: Find the relative extremes of $f(x)=\frac{1}{2}x-\sin x$ on the interval $(0,2\pi)$. Where is it increasing and decreasing on that interval?

Homework Qs Monday, March 16

3.1/ #13) Find the critical numbers.

$$g(t) = t\sqrt{4-t}, \quad t<3$$

$$g(t) = t\,(4-t)^{1/2}$$

$$g'(t) = t\,\frac{d}{dt}(4-t)^{1/2} + (4-t)^{1/2}\,\frac{d}{dt}(t)$$

$$= t\left(\tfrac{1}{2}\right)(4-t)^{-1/2}(-1) + (4-t)^{1/2}(1)$$

$$= -\frac{t}{2\sqrt{4-t}} + \frac{\sqrt{4-t}}{1}\left(\frac{2\sqrt{4-t}}{2\sqrt{4-t}}\right)$$

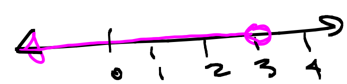
$$= -\frac{t}{2\sqrt{4-t}} + \frac{2(4-t)}{2\sqrt{4-t}}$$

$$= \frac{-t+8-2t}{2\sqrt{4-t}} = \frac{-3t+8}{2\sqrt{4-t}}$$

Where is $g'(t)$ undefined? Set $4-t>0$ to find where it is defined

$$4>t$$
$$t<4$$

Original problem specified domain as $t<3$.



Domain: $(-\infty, 3)$.

$g'(t)$ is defined everywhere on $(-\infty, 3)$.

Where is $g'(t)=0$? Set numerator equal to 0:

$$-3t+8=0$$
$$8=3t$$
$$\frac{8}{3}=t$$
$$t=2\tfrac{2}{3}$$

. This is in the domain

$$\boxed{\text{So } \frac{8}{3} \text{ is the only critical number.}}$$

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$$g(t) = \frac{t^2}{t^2+3}$$. Find absolute extrema on $[-1,1]$.

$$g'(t) = \frac{(t^2+3)(2t)-t^2(2t)}{(t^2+3)^2} = \frac{2t^3+6t-2t^3}{(t^2+3)^2} = \frac{6t}{(t^2+3)^2}$$

Find critical numbers: Denominator is never 0.

Numerator is 0 for $t=0$.

$t=0$ is only critical number.

$$g(-1) = \frac{(-1)^2}{(-1)^2+3} = \frac{1}{4}$$

$$g(1) = \frac{1^2}{1^2+3} = \frac{1}{4}$$

$$g(0) = \frac{0^2}{0^2+3} = 0$$

Absolute max:

$g(1) = g(-1) = \frac{1}{4}$

Absolute min:

$g(0) = 0$.

Would it be correct to write: The absolute ~~max is~~ maxima are $(1, \frac{1}{4})$ and $(-1, \frac{1}{4})$.

No, not correct. The max and min are values (numbers). Ordered pairs are points (locations) on the x-y plane.