

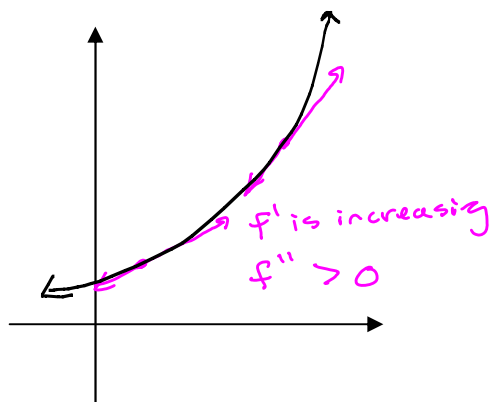
3.4: Concavity and the Second Derivative Test

Concavity:

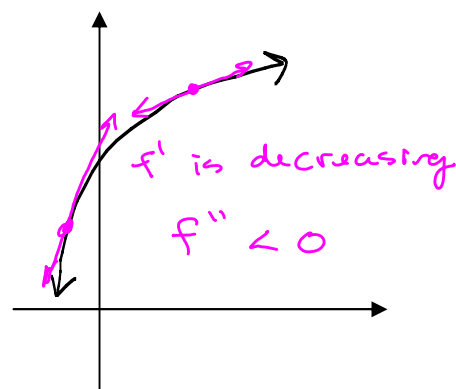
Definition:

- If the graph of f lies above all of its tangents on an interval, then it is called concave upward on that interval. *(graph lies below the secant line)*
- If the graph of f lies below all its tangents on an interval, it is called concave downward on that interval. *(graph lies above the secant line)*

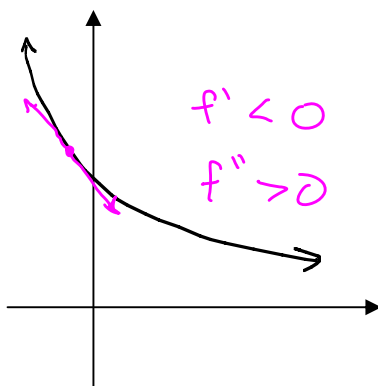
Illustration:



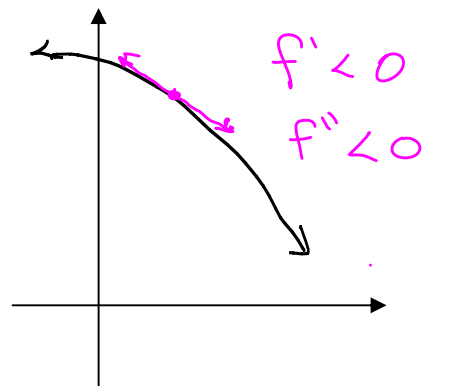
Increasing and Concave Up



Increasing and Concave Down



Decreasing and Concave Up



Decreasing and Concave Down

Notice the slopes of the tangent lines. When the curve is concave up, the slopes are increasing as you move from left to right.

When the curve is concave down, the slopes are decreasing as you move from left to right.

We find out whether f' is increasing or decreasing by looking at its derivative, which is f'' .

Concavity Test:

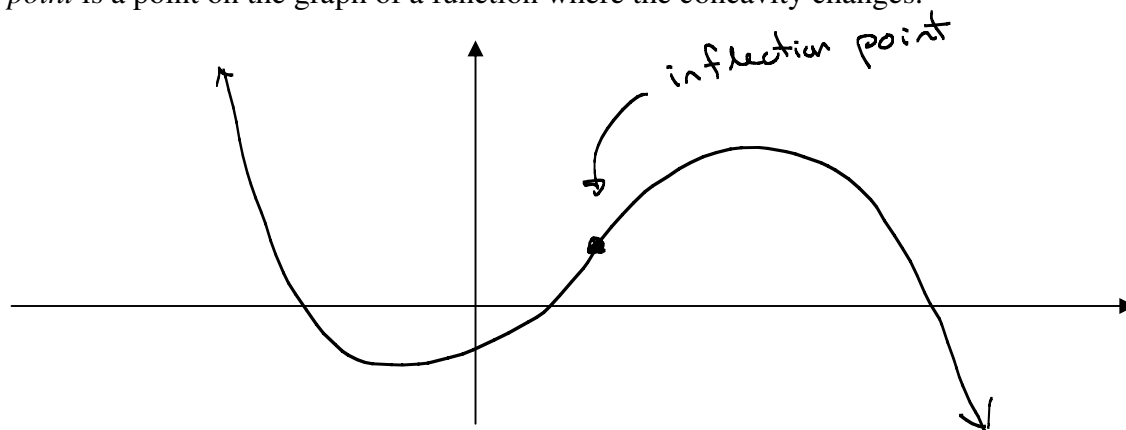
- If $f''(x) > 0$ for all x in (a, b) , then f is concave up on (a, b) .
- If $f''(x) < 0$ for all x in (a, b) , then f is concave down on (a, b) .

Process for Determining Intervals of Concavity:

1. Find the values of x where $f''(x) = 0$ or where $f''(x)$ is not defined. Use these values of x to divide the number line into intervals.
2. Choose a test number c in each interval.
 - If $f''(c) > 0$, then f is concave up on that interval.
 - If $f''(c) < 0$, then f is concave down on that interval.

Inflection points:

An *inflection point* is a point on the graph of a function where the concavity changes.

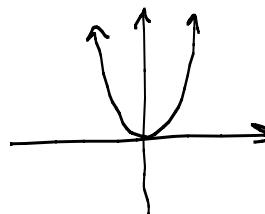
Example 1:

Example 2: Find the intervals on which $f(x) = x^2$ is concave up and concave down.

$$f'(x) = 2x$$

$$f''(x) = 2 > 0$$

concave up on $(-\infty, \infty)$
No inflection points



Example 3: Determine the intervals of concavity and the inflection points of

$$f(x) = x^3 + 6x^2 - 36x + 18.$$

$$f'(x) = 3x^2 + 12x - 36$$

$$f''(x) = 6x + 12 = 6(x+2)$$

$$f''(x) = 0 \text{ for } x = -2.$$

 $(-\infty, -2)$: Test number $x = -3$

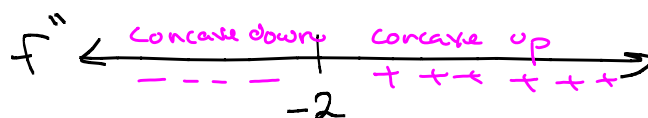
$$f''(-3) = 6(-3) + 12 = -18 + 12 = -6$$

(-)
concave down

 $(2, \infty)$: Test number $x = 3$

$$f''(3) = 6(3) + 12 > 0 \quad (+)$$

concave up



Concave down on $(-\infty, -2)$.

Concave up on $(2, \infty)$.

Inflection point $(-2, 106)$

Inflection pt at $x = -2$

Find y-value:

$$f(-2) = (-2)^3 + 6(-2)^2 - 36(-2) + 18 = 106$$

Example 4: Determine the intervals of concavity and the inflection points of $f(x) = x + \frac{4}{x}$.

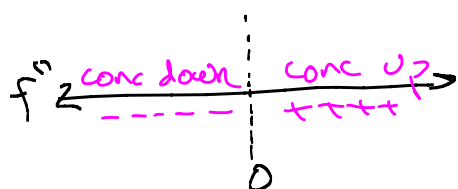
$$f(x) = x + 4x^{-1}$$

$$f'(x) = 1 - 4x^{-2}$$

$$f''(x) = 8x^{-3} = \frac{8}{x^3}$$

$f''(x)$ is never 0 (because numerator is never 0)

$f''(x)$ is undefined at $x = 0$.


 $(-\infty, 0)$: Test $x = -1$

$$f''(-1) = \frac{8}{(-1)^3} = -8 < 0 \quad \text{concave down}$$

 $(0, \infty)$: Test $x = 1$

$$f''(1) = \frac{8}{1^3} = 8 > 0 \quad \text{concave up}$$

Concave up on $(0, \infty)$.
Concave down on $(-\infty, 0)$.
No inflection points.



3.4.4

The second derivative test:

Notice: For a smooth (differentiable) function, the graph is concave upward at a relative minimum and concave downward at a relative maximum.

Therefore, at a critical number, we can look at the sign of f'' to determine whether there is a relative minimum or relative maximum at that critical number.

The Second Derivative Test (for Local Extremes):

Suppose f'' is continuous near c .

- If $f'(x) = 0$ and $f''(c) < 0$, then f has a relative maximum at c .
- If $f'(x) = 0$ and $f''(c) > 0$, then f has a relative minimum at c .
- If $f'(x) = 0$ and $f''(c) = 0$, then the test is inconclusive. Use the 1st derivative test instead.

Example 5: Use the second derivative test to find the local extremes of $f(x) = x^3 + 6x^2 - 36x + 18$.

$$\begin{aligned} f'(x) &= 3x^2 + 12x - 36 \\ &= 3(x^2 + 4x - 12) \\ &= 3(x+6)(x-2) \end{aligned}$$

critical numbers: $-6, 2$

$$\begin{aligned} f''(x) &= 6x + 12 \\ f''(-6) &= 6(-6) + 12 = -36 + 12 = -24 < 0 \quad \text{relative max at } x = -6 \\ f''(2) &= 6(2) + 12 > 0 \quad \text{concave up, so relative min at } x = 2 \end{aligned}$$

(should find y-values too)

Example 6: Determine the local extremes of $f(x) = -2x^4 + 4x^3$

$$\begin{aligned} f'(x) &= -8x^3 + 12x^2 \\ &= -4x^2(2x - 3) \end{aligned}$$

$$x = 0 \mid 2x - 3 = 0$$

$$2x = 3$$

$$x = \frac{3}{2}$$

Critical numbers: $0, \frac{3}{2}$

2nd derivative test:

$$\begin{aligned} f''(x) &= -24x^2 + 24x \\ &= -24x(x - 1) \end{aligned}$$

$$f''(0) = -24(0)^2 + 24(0) = 0$$

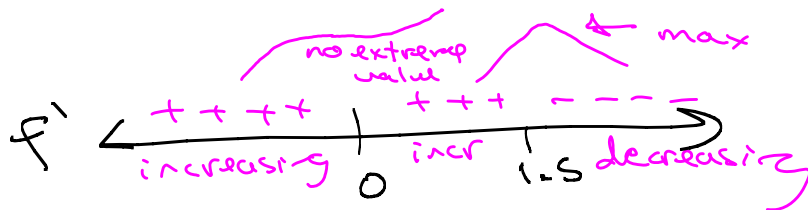
2nd derivative test is inconclusive
Use 1st derivative test.

See next page

Ex 6 cont'd:

$$f'(x) = -4x^2(2x-3)$$

critical #s: $0, \frac{3}{2} = 1.5$



$(-\infty, 0)$: Test $x = -1$

$$\begin{aligned} f'(-1) &= -4(-1)^2(2(-1)-3) \\ &= -4(1)(-2-3) \\ &= -4(-5) \\ &= (+) \end{aligned}$$

$(1.5, \infty)$: Test $x = 2$

$$\begin{aligned} f'(2) &= -4(2)^2(2 \cdot 2 - 3) \\ &= -4(4)(1) \\ &= (-) \end{aligned}$$

$(0, 1.5)$: Test $x = 1$

$$\begin{aligned} f'(1) &= -4(1)^2(2(1)-3) \\ &= (-4)(-1) \\ &= + \quad (+) \end{aligned}$$

Relative max:

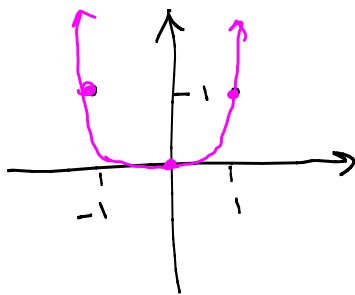
$$f(1.5) = \frac{27}{8}$$

$$\begin{aligned} f(1.5) &= -2\left(\frac{3}{2}\right)^4 + 4\left(\frac{3}{2}\right)^3 \\ &= -2\left(\frac{81}{16}\right) + 4\left(\frac{27}{8}\right) \\ &= -\frac{81}{8} + \frac{108}{8} = \frac{27}{8} \end{aligned}$$

Ex: $f(x) = x^6$

$$f'(x) = 6x^5$$

critical number: 0



$$f''(x) = 30x^4$$

2nd derivative test: $f''(0) = 30(0)^4 = 0$

inconclusive

1st derivative test.

f' $\leftarrow \text{---} \mid \text{---} \rightarrow$

decr 0 incr

relative min:

$$f(0) = 0$$