3.5: Limits at Infinity

There are two types of limits involving infinity.

Limits at infinity, written in the form $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$, are related to horizontal asymptotes.

Infinite limits (covered in Section 1.5) take the form of statements like $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. Infinite limits generally result in vertical asymptotes.

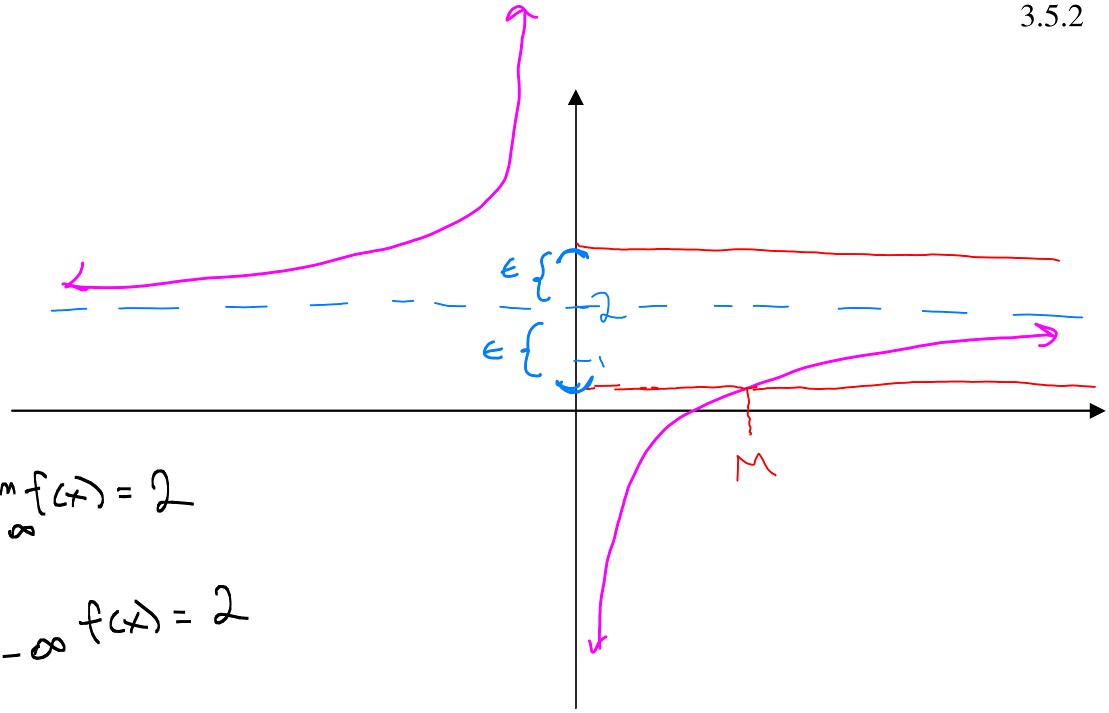
When combined, these two types of limits involving infinity result in statements such as $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = -\infty$, which describe the end behavior of graphs.

Limits at infinity:

> Let *f* be a function defined on some interval (a, ∞). Then
>
> $$\lim_{x\to\infty} f(x) = L$$
>
> means that the values of $f(x)$ can be made arbitrarily close to *L* by taking *x* sufficiently large.
>
> More precisely, $\lim_{x\to\infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number $M > 0$ such that for all *x*, $|f(x) - L| < \varepsilon$ whenever $x > M$.

> Let *f* be a function defined on some interval $(-\infty, a)$. Then
>
> $$\lim_{x\to-\infty} f(x) = L$$
>
> means that the values of $f(x)$ can be made arbitrarily close to *L* by making *x* a sufficiently large negative number.
>
> More precisely, $\lim_{x\to\infty} f(x) = L$ if, for every number $\varepsilon > 0$, there exists a corresponding number
> $x \to -\infty$
> $N < 0$ such that for all *x*, $|f(x) - L| < \varepsilon$ whenever $x < N$.

Example 1:



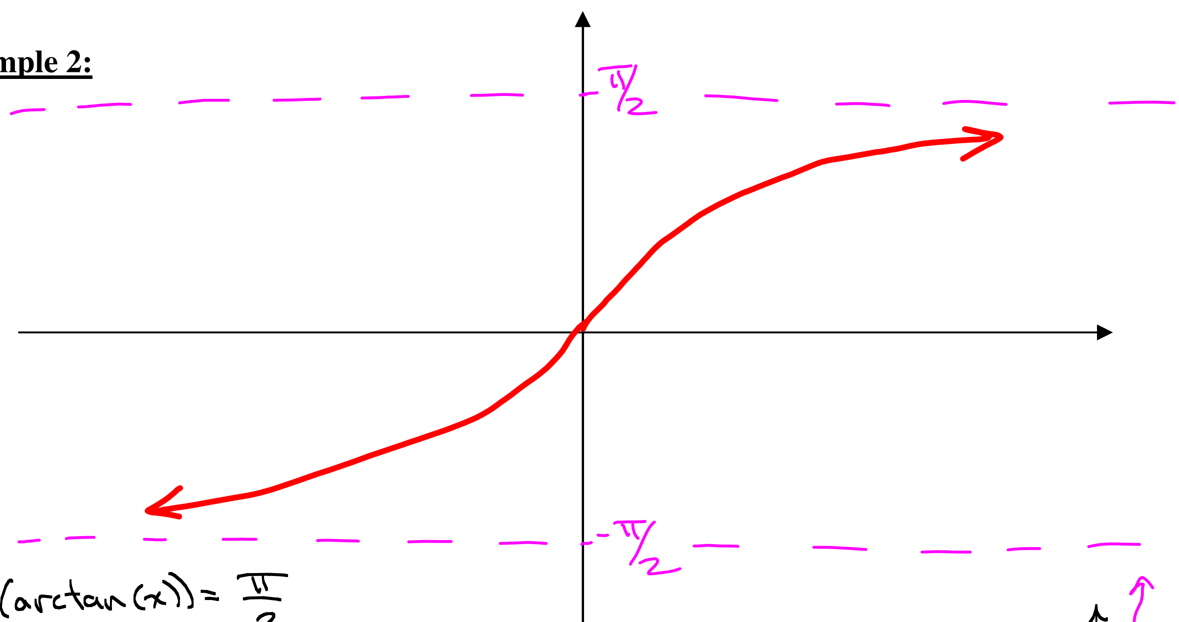
$$\lim_{x\to\infty} f(x) = 2$$

$$\lim_{x\to -\infty} f(x) = 2$$

Horizontal asymptotes:

> The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either
>
> $$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

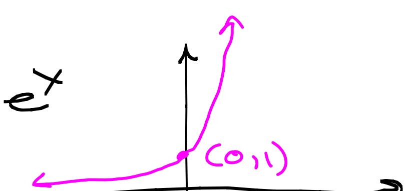
Example 2:



$$\lim_{x\to\infty} (\arctan(x)) = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} (\arctan(x)) = -\frac{\pi}{2}$$

Ex: $f(x) = e^x$



$$\lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to+\infty} e^x = +\infty \quad \text{(limit does not exist)}$$

Example 3: Determine $\lim_{x\to\infty}\frac{1}{x}$ and $\lim_{x\to-\infty}\frac{1}{x}$.

As $x \longrightarrow +\infty$, $\frac{1}{x} \longrightarrow \frac{1}{+\text{huge}} \longrightarrow +\text{tiny} \longrightarrow 0$.

As $x \longrightarrow -\infty$, $\frac{1}{x} \longrightarrow \frac{1}{-\text{huge}} \longrightarrow -\text{tiny} \longrightarrow 0$

$$\boxed{\begin{array}{l} \lim_{x\to\infty}\frac{1}{x} = 0 \\ \lim_{x\to-\infty}\frac{1}{x} = 0 \end{array}}$$

Sometimes we write

$$\lim_{x\to\pm\infty}\frac{1}{x} = 0$$

Theorem: If $r > 0$ is a rational number, then $\lim_{x\to\infty}\frac{1}{x^r} = 0$. Also $\lim_{x\to-\infty}\frac{1}{x^r} = 0$ if x^r is defined for all x. (Also true for r irrational)

$\lim_{x\to\infty}\frac{1}{x^r} = 0$ and $\lim_{x\to-\infty}\frac{1}{x^r} = 0$

Example 4: Determine $\lim_{x\to\infty}\left(9+\frac{3}{x^4}\right)$

$$\lim_{x\to\infty}\left(9+\frac{3}{x^4}\right) = \lim_{x\to\infty} 9 + \lim_{x\to\infty}\frac{3}{x^4} = 9+0 = \boxed{9}$$

To take advantage of this theorem when determining the limit at infinity of a quotient, divide numerator and denominator by the largest power of the variable in the denominator.

Example 5: Find $\lim_{x\to\infty}\frac{3-2x}{4x+6}$.

$$\lim_{x\to\infty}\frac{3-2x}{4x+6} = \lim_{x\to\infty}\frac{3-2x}{4x+6}\left(\frac{1/x}{1/x}\right) = \lim_{x\to\infty}\left(\frac{\frac{3}{x}-\frac{2x}{x}}{\frac{4x}{x}+\frac{6}{x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\frac{3}{x}-2}{4+\frac{6}{x}}\right) = \frac{0-2}{4+0} = \frac{-2}{4} = \boxed{-\frac{1}{2}}$$

Example 6: Find $\lim_{x\to\infty}\frac{2x^2+15x+9}{5x^3-14}$ and $\lim_{x\to-\infty}\frac{2x^2+15x+9}{5x^3-14}$.

$$\lim_{x\to\infty}\frac{2x^2+15x+9}{5x^3-14} = \lim_{x\to\infty}\frac{\frac{2x^2}{x^3}+\frac{15x}{x^3}+\frac{9}{x^3}}{\frac{5x^3}{x^3}-\frac{14}{x^3}} = \lim_{x\to\infty}\frac{\frac{2}{x}+\frac{15}{x^2}+\frac{9}{x^3}}{5-\frac{14}{x^3}}$$

$$= \frac{0+0+0}{5-0} = \frac{0}{5} = \boxed{0}$$

$$\lim_{x\to-\infty}\frac{2x^2+15x+9}{5x^3-14} = \lim_{x\to-\infty}\frac{\frac{2}{x}+\frac{15}{x^2}+\frac{9}{x^3}}{5-\frac{14}{x^3}} = \frac{0+0+0}{5-0} = \boxed{0}$$

Note: As $x\to\pm\infty$, $\frac{2x^2+15x+9}{5x^3-14} \longrightarrow \frac{2x^2}{5x^3} \longrightarrow \frac{2}{5x} \to 0$

Example 7: Find the horizontal asymptote (if any) of $h(x)=\frac{8x^2-6x+1}{3x^2+4x-5}$.

$$\lim_{x\to\pm\infty}\frac{\frac{8x^2}{x^2}-\frac{6x}{x^2}+\frac{1}{x^2}}{\frac{3x^2}{x^2}+\frac{4x}{x^2}-\frac{5}{x^2}} = \lim_{x\to\pm\infty}\frac{8-\frac{6}{x}+\frac{1}{x^2}}{3+\frac{4}{x}-\frac{5}{x^2}}$$

$$= \frac{8-0+0}{3+0-0} = \frac{8}{3}$$

The horizontal asymptote is $y=\frac{8}{3}$.

Example 8: Determine $\lim_{x\to\infty}\frac{7x^5-5x+1}{8-x^2}$ and $\lim_{x\to-\infty}\frac{7x^5-5x+1}{8-x^2}$.

$$\lim_{x\to\infty}\frac{7x^5-5x+1}{8-x^2} = \lim_{x\to\infty}\frac{\frac{7x^5}{x^2}-\frac{5x}{x^2}+\frac{1}{x^2}}{\frac{8}{x^2}-\frac{x^2}{x^2}} = \lim_{x\to\infty}\frac{7x^3-\frac{5}{x}+\frac{1}{x^2}}{\frac{8}{x^2}-1}$$

$$= \frac{\lim_{x\to\infty}(7x^3)-0+0}{0-1} = \frac{\lim_{x\to\infty}(7x^3)}{-1} = \boxed{-\infty}$$

(limit does not exist)

$$\lim_{x\to-\infty}\frac{7x^5-5x+1}{8-x^2} = \frac{\lim_{x\to-\infty}(7x^3)}{-1} = \boxed{+\infty}$$

(limit does not exist)

Scratchwork

As $x\to+\infty$

$\frac{7x^3}{-1} \to \frac{7(+\text{huge})^3}{-1}$

$\to -7(+\text{huge})$

$\to -\text{huge}$

As $x\to-\infty$, $\frac{7x^3}{-1} = -7x^3 \to -7(-\text{huge})^3$

$\to -7(-\text{huge}) \to +\text{huge}$

Note: $\infty - \infty$ is an indeterminate form (as $\frac{0}{0}$ is an indeterminate form)

Example 9: Determine $\lim\limits_{x\to\infty} \dfrac{\sqrt[3]{x}-\sqrt{x}}{8-x^{-2}}$

$$\lim_{x\to\infty} \frac{\sqrt[3]{x}-\sqrt{x}}{8-x^{-2}} = \lim_{x\to\infty} \frac{\sqrt[3]{x}-\sqrt{x}}{8-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^{\frac{1}{3}}-x^{\frac{1}{2}}}{8-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{x^{\frac{1}{3}}\left(1-x^{\frac{1}{2}-\frac{1}{3}}\right)}{8-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{x^{\frac{1}{3}}\left(1-x^{\frac{1}{6}}\right)}{8-\frac{1}{x^2}} = \boxed{-\infty}$$

(limit does not exist)

Scratchwork

As $x \to +\infty$, $\dfrac{x^{\frac{1}{3}}(1-x^{1/6})}{8-\frac{1}{x^2}} \to \dfrac{\sqrt[3]{+\text{huge}}\,(1-\sqrt[6]{+\text{huge}})}{8-0} \to \dfrac{+\text{huge}\,(-\text{huge})}{8} \to -\dfrac{\text{huge}}{8} \to -\text{huge}$

Example 10: Evaluate the limit of $f(x) = \dfrac{x^3+4x^2-11x-7}{x^2+6x+1}$ as x approaches $\pm\infty$.

$$\lim_{x\to\infty} \frac{x^3+4x^2-11x-7}{x^2+6x+1} = \lim_{x\to\infty} \frac{\frac{x^3}{x^2}+\frac{4x^2}{x^2}-\frac{11x}{x^2}-\frac{7}{x^2}}{\frac{x^2}{x^2}+\frac{6x}{x^2}+\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{x+4-\frac{11}{x}-\frac{7}{x^2}}{1+\frac{6}{x}+\frac{1}{x^2}} = \frac{\lim\limits_{x\to\infty}(x)+4-0-0}{1+0+0} = \boxed{+\infty}$$

Scratch

As $x\to\infty$

$x+4 \to$ +huge $+4$

$\to$ +huge

$$\lim_{x\to-\infty} \frac{x^3+4x^2-11x-7}{x^2+6x+1} = \frac{\lim\limits_{x\to-\infty}(x)+4}{1} = \boxed{-\infty}$$

(limits don't exist)

Scratchwork

As $x\to-\infty$, $x+4 \to$ $-$huge$+4 \to -$huge

Example 11: Evaluate the limit of $f(x) = \dfrac{8x^2-7x+1}{2x-3}$ as x approaches $\pm\infty$.

$$\lim_{x\to\infty} \frac{8x^2-7x+1}{2x-3} = \lim_{x\to\infty} \frac{8x-7+\frac{1}{x}}{2-\frac{3}{x}} = \frac{\lim\limits_{x\to\infty}(8x)-7+0}{2-0} = \boxed{+\infty}$$

$$\lim_{x\to-\infty} \frac{8x^2-7x+1}{2x-3} = \frac{\lim\limits_{x\to-\infty}(8x)-7+0}{2-0} = \boxed{-\infty}$$

(limits do not exist)

Problem 11b: Does f have a slant (oblique) asymptote? Find it.

Long Division

$\frac{8x^2}{2x}$ $\quad$ $\frac{5x}{2x}$

$$\begin{array}{r} 4x + \frac{5}{2} \\ 2x-3 \overline{)\, 8x^2-7x+1} \\ -(8x^2-12x) \\ \hline 5x+1 \\ -(5x-\frac{15}{2}) \\ \hline \frac{17}{2} \end{array}$$

See next page...

Problem 11 cont'd

$$f(x) = \frac{8x^2-7x+1}{2x-3} = 4x + \frac{5}{2} + \frac{17/2}{2x-3}$$

Note:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(4x+\frac{5}{2}+\frac{17/2}{2x-3}\right)$$

this term approaches 0 as $x \to \pm\infty$

The graph of f has a slant (oblique) asymptote: $\boxed{y = 4x+\frac{5}{2}}$

Ex 10: $f(x) = \dfrac{x^3+4x^2-11x-7}{x^2+6x+1}$ has a slant asymptote also.

$$\begin{array}{r l}
 & x \quad -2 \\
x^2+6x+1 \,\big)\!\! & \overline{x^3 + 4x^2 - 11x - 7} \\
 & -(x^3 + 6x^2 + x) \\
\hline
 & -2x^2 - 12x - 7 \\
 & -(-2x^2 - 12x - 2) \\
\hline
 & -5
\end{array}$$

So $f(x) = x-2+\dfrac{-5}{x^2+6x+1}$

As $x \to \pm\infty$, this term $\to 0$

As $x \to \pm\infty$,
$f(x) \to x-2$

Slant asymptote: $\boxed{y = x-2}$

Slant asymptotes:

> Note:
> The graphs of the functions in the previous two examples have *oblique (slant) asymptotes*. This is because the function values (y-values) approached those of a linear function $y = mx + b$ as x approached $\pm\infty$.

Example 12: Evaluate the limit of $f(x) = \dfrac{2x^3 - x^2 + x}{x-3}$ as x approaches $\pm\infty$. Does the graph of this function have a slant asymptote?

$$\begin{array}{r|l} & 2x^2 + 5x + 16 \\ x-3 & 2x^3 - x^2 + x + 0 \\ & -(2x^3 - 6x^2) \\ & \quad 5x^2 + x \\ & \quad -(5x^2 - 15x) \\ & \qquad 16x + 0 \\ & \qquad -(16x - 48) \\ & \qquad\qquad 48 \end{array}$$

$f(x) = 2x^2 + 5x + 16 + \dfrac{48}{x-3}$

$$\boxed{\lim_{x\to\infty} f(x) = +\infty}$$

Scratchwork

As $x \to \infty$, $f(x) \to 2(+\text{huge})^2 \to +\infty$

As $x \to -\infty$, $f(x) \to 2(-\text{huge})^2 \to 2(+\text{huge}) \to +\infty$

$$\boxed{\lim_{x\to-\infty} f(x) = +\infty}$$

Graph it:

f is asymptotic to $y = 2x^2 + 5x + 16$

$y = 2\left(x^2 + \frac{5}{2}x\right) + 16$

$y = 2\left(x^2 + \frac{5}{2}x + \frac{25}{16}\right) + 16 - 2\left(\frac{25}{16}\right)$

$\left(\frac{5}{2} \cdot \frac{1}{2}\right)^2 = \left(\frac{5}{4}\right)^2 = \frac{25}{16}$

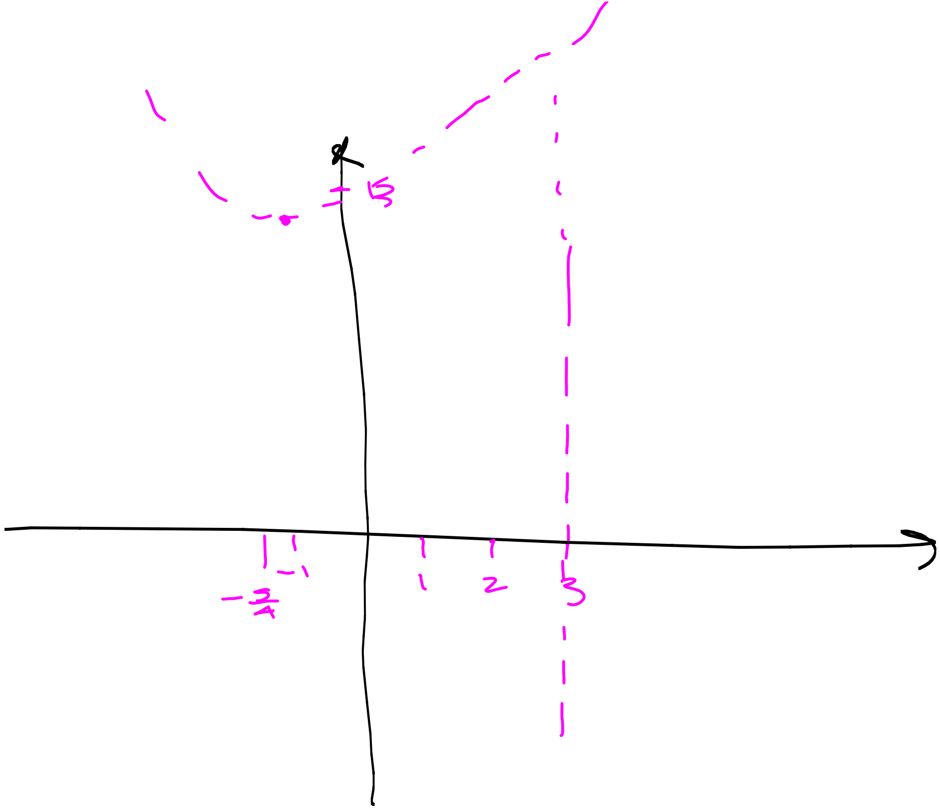
$y = 2\left(x + \frac{5}{4}\right)^2 + 16 - \frac{25}{8}$

$y = 2\left(x + \frac{5}{4}\right)^2 + \frac{103}{8}$

$$\begin{array}{r|l} & 12 \\ 8 & 103 \\ & 8 \\ & 23 \\ & 16 \\ & 7 \end{array}$$

$$\begin{array}{r} 16 \\ \times 8 \\ \hline 128 \\ -25 \\ \hline 103 \end{array}$$

Graph next page



15
$-\frac{3}{4}$
1
2
3