3.6: A Summary of Curve Sketching

Steps for Curve Sketching

1. Determine the domain of f.
2. Find the x-intercepts and y-intercept, if any.
3. Determine the "end behavior" of f, that is, the behavior for large values of $|x|$ (limits at infinity). (often this is optional)
4. Find the vertical, horizontal, and oblique asymptotes, if any. → figure out if it crosses the horizontal asymptote
5. Determine the intervals where f is increasing/decreasing.
6. Find the relative extremes of f, if any. (You should find both the x- and y-values.)
7. Determine the intervals where f is concave up/concave down.
8. Find the inflection points, if any. (You should find both the x- and y-values.)
9. Plot more points if necessary, and sketch the graph.

Example 1: Sketch the graph of $f(x) = x^3 - 6x^2 + 9x$.

Domain: $(-\infty, \infty)$

$f(x) = x(x^2 - 6x + 9)$

$= x(x-3)^2$

Find x-intercepts:

set $y=0$: $0 = x(x-3)^2$

$x = 0, 3$

Find y-intercept: set $x=0$

$y = f(0) = 0^3 - 6(0)^2 + 9(0) = 0$

x-intercepts: $(0,0)$, $(3,0)$

y-intercept: $(0,0)$

Relative max at: $(1,4)$

Relative min at: $(3,0)$

Inflection pt: $(2,2)$

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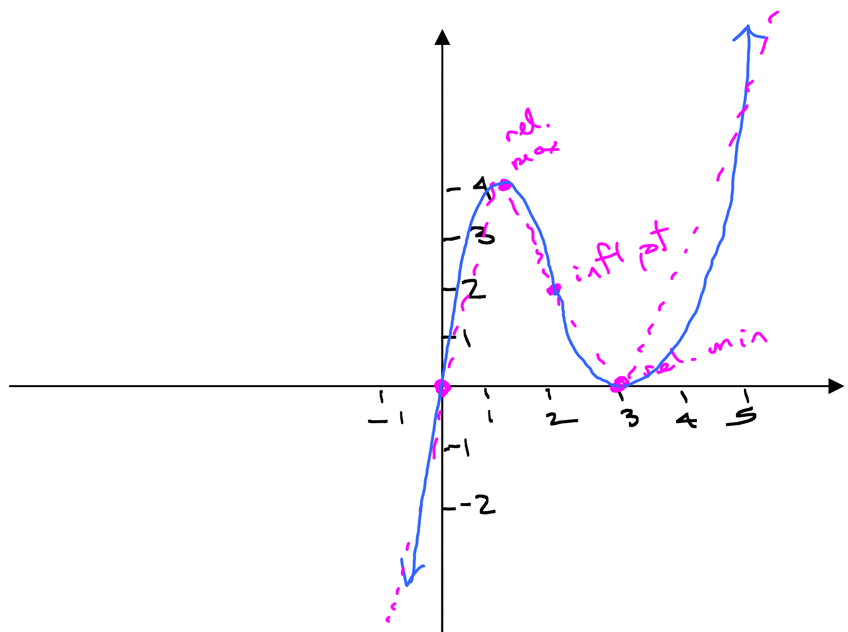
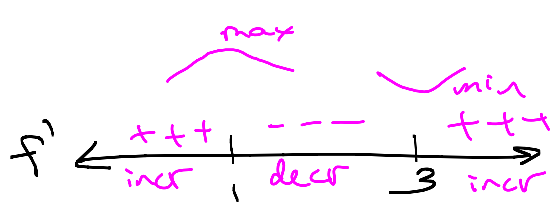
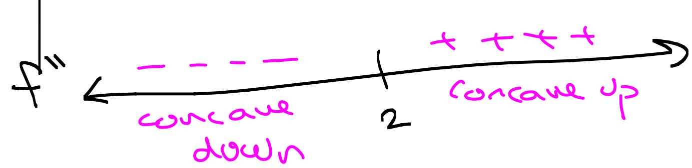






Ex 1 cont'd:

$f(x) = x^3 - 6x^2 + 9x$

$f'(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x-3)(x-1)$

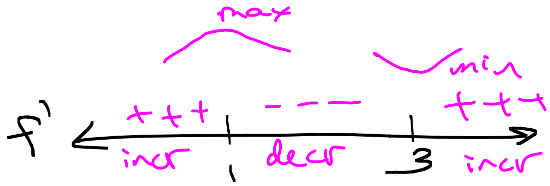
Critical numbers: 3, 1

$(-\infty, 1)$: Test $x = 0$

$f'(0) = 3(0-3)(0-1) \Rightarrow (+)$

$(1, 3)$: Test $x = 2$

$(3, \infty)$: Test $x = 4$



Relative max: $f(1) = 4$

$f(1) = 1^3 - 6(1)^2 + 9(1)$

$= 1 - 6 + 9 = 4$

Relative min: $f(3) =$

$f(3) = 3^3 - 6(3)^2 + 9(3)$

$= 27 - 54 + 27 = 0$

$f''(x) = 6x - 12$

$= 6(x-2)$

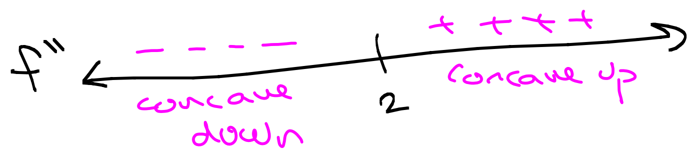
$f''(x) = 0$ for $x = 2$

$(-\infty, 2)$: Test $x = 0$

$f''(0) = 6(0-2) = -12 < 0$ concave down

$(2, \infty)$: Test $x = 3$

$f''(3) = 6(3) - 12 = 18 - 12 = 6 > 0$ concave up



Inflection point at $x = 2$.

Find y-value:

$f(2) = (2)^3 - 6(2)^2 + 9(2)$

$= 8 - 24 + 18$

$= -16 + 18 = 2$

Example 2: Sketch the graph of $f(x) = 3x^4 + 4x^3$.

$f(x) = x^3(3x+4)$

Find x-intercepts: Set $y=0$: $0 = x^3(3x+4)$

$x=0, \quad x = -\frac{4}{3}$

Find y-intercept: Set $x=0$

$f(0) = 3(0)^4 + 4(0)^3 = 0$

Find increasing/decreasing intervals

$f'(x) = 12x^3 + 12x^2$

$= 12x^2(x+1)$

critical #s: $0, -1$

$(-\infty, -1)$: Test $x=-2$

$f'(-2) = 12(-2)^2(-2+1)$

$\Rightarrow (+)(-) \Rightarrow (-)$

$(-1, 0)$: Test $x = -0.5$

$f'(-0.5) = 12(-0.5)^2(-0.5+1)$

$\Rightarrow (+)(+)(+) \Rightarrow (+)$

$(0, \infty)$: Test $x=1$

$f'(1) = 12(1)^2(1+1)$

$\Rightarrow (+)$

Relative min at $x=-1$:

Find y-value:

$f(-1) = 3(-1)^4 + 4(-1)^3$

$= 3 - 4 = -1$

Find concavity:

$f''(x) = 36x^2 + 24x$

$= 12x(3x+2)$

$f''(x) = 0$ for $x=0, x=-\frac{2}{3}$

$(-\infty, -\frac{2}{3})$: Test $x=-1$

$f''(-1) = 12(-1)(3(-1)+2)$

$= (-)(-) \Rightarrow (+)$

x-intercepts: $(0,0), (-\frac{4}{3}, 0)$

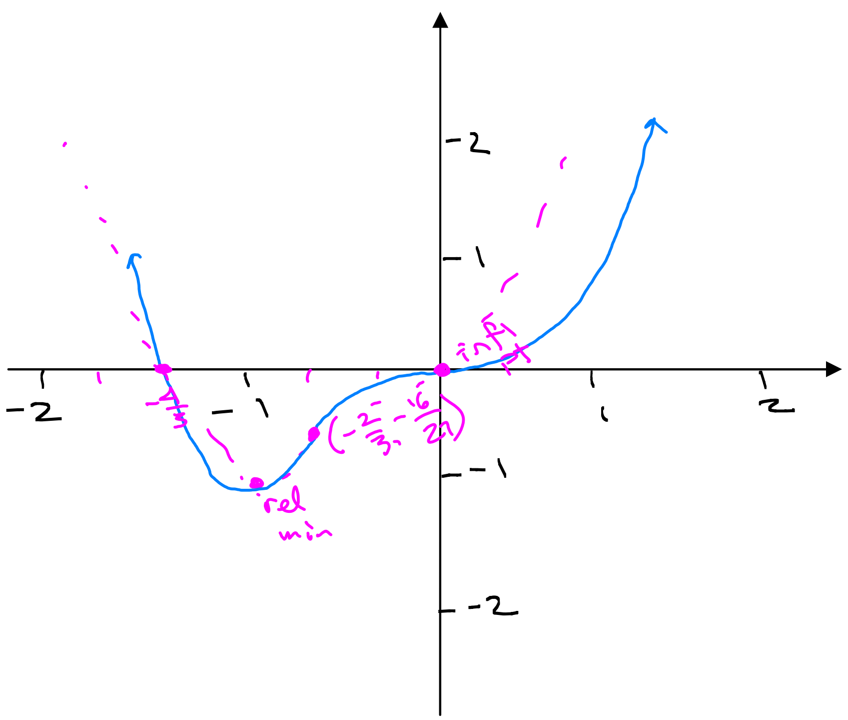
y-intercept: $(0,0)$

Relative min: $(-1,-1)$

Inflection pts: $(0,0)$, $(-\frac{2}{3}, -\frac{16}{27})$

f': − − − (decr) | -1 | + + + (incr) | 0 | + + + (incr)

f'': + + + (conc up) | $-\frac{2}{3}$ | − − − (conc down) | 0 | + + + + (conc up)



Inflection points at $x=0, x=-\frac{2}{3}$

$f(-\frac{2}{3}) = -\frac{16}{27}$

$(-\frac{2}{3}, 0)$: Test $x = -\frac{1}{3}$

$f''(-\frac{1}{3}) = 12(-\frac{1}{3})(3(-\frac{1}{3})+2) = -4(1) \Rightarrow (-)$

Example 3: Sketch the graph of $f(x) = \frac{2x}{x^2 - 1}$.

$$f(x) = \frac{2x}{(x+1)(x-1)}$$

x-intercepts where numerator is 0.
vertical asymptotes where denominator is 0.

x-intercept: 0 so (0,0)
vertical asymptotes: $x = -1$, $x = 1$
Find y-intercept: set $x = 0$

$$f(0) = \frac{2(0)}{0^2 - 1} = 0$$

As $x \to \pm\infty$, $y \to \frac{2x}{x^2} = \frac{2}{x} \to 0$

Horizontal asymptote: $y = 0$

$$f'(x) = \frac{-2(x^2+1)}{(x^2-1)^2}$$

$$f''(x) = \frac{4x(x^2+3)}{(x^2-1)^3}$$

f': − − − decr | −1 | − − − − decr | 1 | − − − decr

f'': − − − − conc down | −1 | + + conc up | 0 | − − conc down | 1 | + + + conc up

$f'(x)$ is never 0. (numerator can't be 0)
$f'(x)$ is undefined at $x = \pm 1$ (same places as original function)
$f'(x) \Rightarrow \frac{(-2)(+)}{(+)} \Rightarrow (-)$ everywhere it's defined

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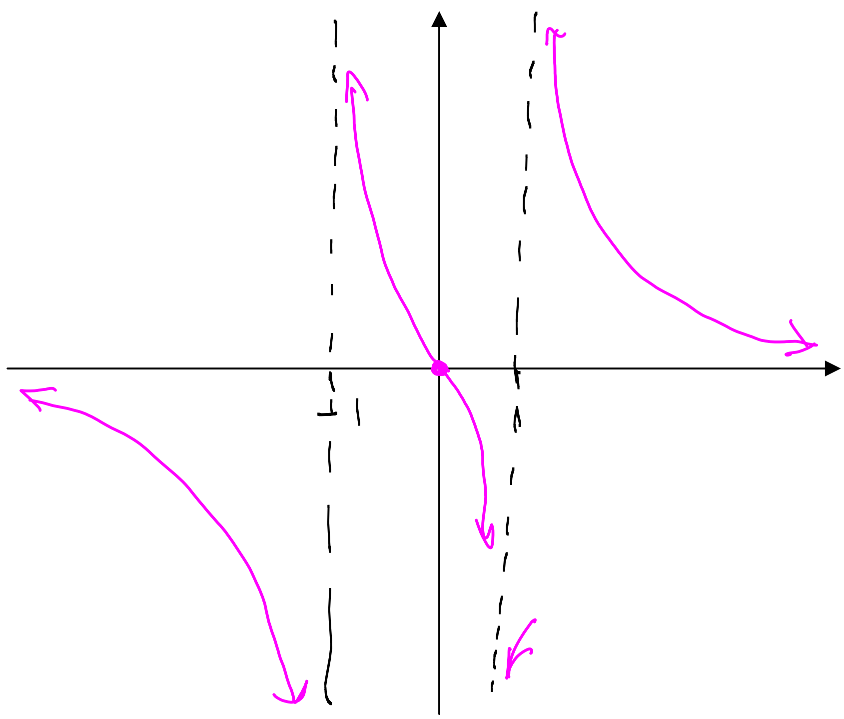
x-intercept: (0,0)
vertical asymptotes: $x = \pm 1$
y-intercept: (0,0)
horizontal asymptote: $y = 0$



f': − − − decr | −1 | − − − − decr | 1 | − − − decr

f'': − − − − conc down | −1 | + + conc up | 0 | − − conc down | 1 | + + + conc up

Find concavity

$$f''(x) = \frac{4x(x^2+3)}{(x^2-1)^3}$$

undefined for $x = \pm 1$
(same as original fcn)

$f''(x) = 0$ for $x = 0$ from numerator

$(-\infty, -1)$: $f''(-2) = \frac{4(-2)(+)}{((-2)^2-1)^3} \Rightarrow \frac{(-)(+)}{(4-1)^3} \Rightarrow \frac{(-)}{(+)} \Rightarrow (-)$

Prev example:

$(-1,0)$: Test $x=-\frac{1}{2}$

$f''(-\frac{1}{2}) \Rightarrow \dfrac{4(-\frac{1}{2})(+)}{((-\frac{1}{2})^2-1)^3} \Rightarrow \dfrac{(-)}{(\frac{1}{4}-1)^3} \Rightarrow \dfrac{(-)}{(-)^3} = \dfrac{(-)}{(-)} \Rightarrow (+)$

Example 4: Sketch the graph of $f(x)=\dfrac{x^2+1}{x^2-4}$.

$$f(x)=\frac{x^2+1}{(x+2)(x-2)}$$

Vertical Asymptotes: $x=\pm 2$

Horizontal Asymptotes: $y=1$

x-intercepts: none

y-intercept: $-\frac{1}{4}$

Find horizontal asymptote:

As $x \to \pm\infty$, $y \to \frac{x^2}{x^2}=1$

Horiz. asy. is $y=1$

Find x-intercepts:

Set $y=0$: $0=\dfrac{x^2+1}{x^2-4}$

$0=x^2+1$

$x^2=-1$ no real sol'n

Find y-intercept:

Set $x=0$: $y=\dfrac{0^2+1}{0^2-4}=-\frac{1}{4}$

Does it cross the horizontal asymptote?

Set $y=1$: $1=\dfrac{x^2+1}{x^2-4}$

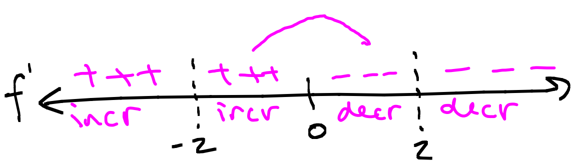
$x^2-4=x^2+1$

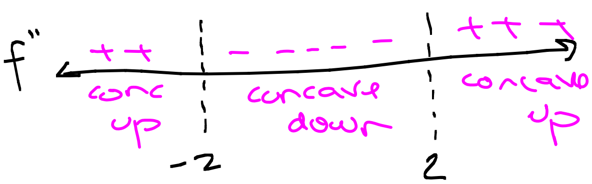
$-4=1$ Always false

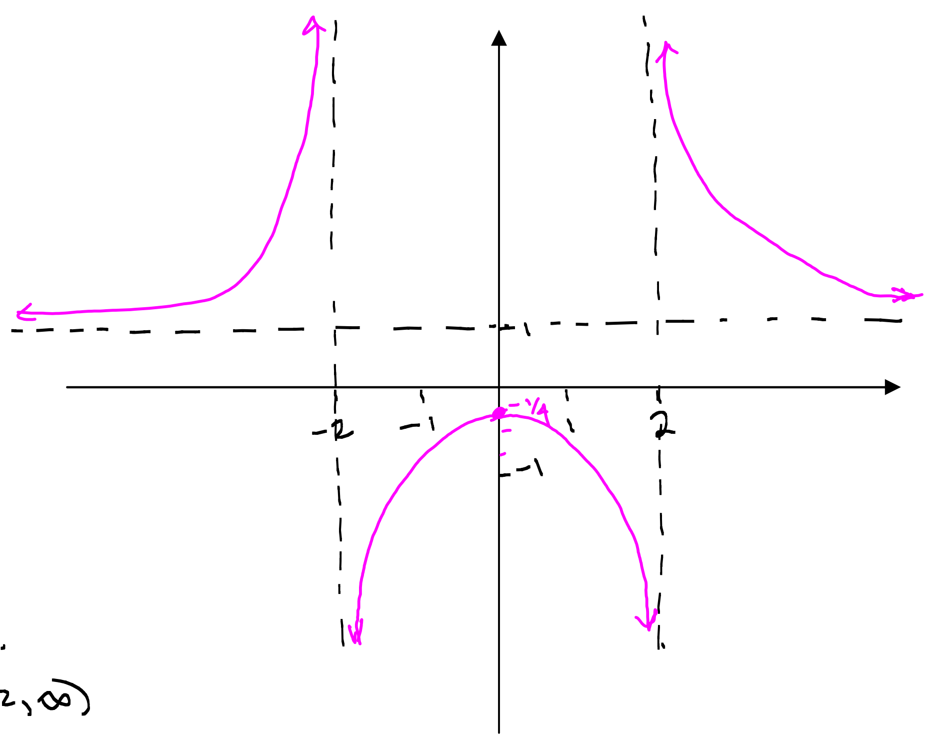
No solutions, so No, it does not cross the horizontal asymptote.

$f'(x)=\dfrac{-10x}{(x^2-4)^2}$ critical number: 0

$f''(x)=\dfrac{10(3x^2+4)}{(x^2-4)^3}$







Relative max at 0:

$f(0)=-\frac{1}{4}$

No inflection points

Increasing on $(-\infty,-2)$ and on $(-2,0)$.

Decreasing on $(0,2)$ and on $(2,\infty)$.

Concave up on $(-\infty,-2)$ and $(2,\infty)$

concave down on $(-2,2)$.

Example 5: Sketch the graph of $f(x) = \dfrac{x^2 - 4}{x+3}$.

$f(x) = \dfrac{(x+2)(x-2)}{x+3}$

x-intercepts: ± 2

y-intercept: $-\frac{4}{3}$

vertical asymptotes: $x = -3$

horizontal asymptote: none

Notice: as $x \to \pm\infty$, $y \to \frac{x^2}{x} \Rightarrow x \to \pm\infty$

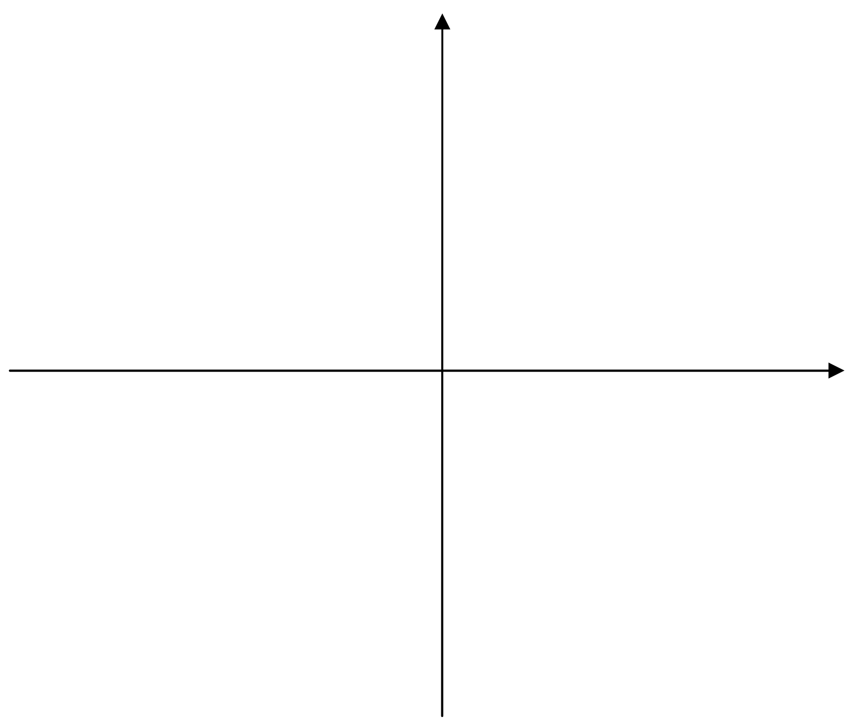
no horizontal asymptote

deg(num) = deg(denom) + 1, so we have a slant asymptote:

$$\begin{array}{r} x - 3 \\ x+3 \enclose{longdiv}{x^2 + 0x - 4} \\ -(x^2 + 3x) \\ \hline -3x - 4 \\ -(-3x - 9) \\ \hline 5 \end{array}$$

$f(x) = \dfrac{x^2 - 4}{x+3} = x - 3 + \dfrac{5}{x+3}$

Slant asymptote: $y = x - 3$



Ex 5 cont'd: $f(x) = \dfrac{x^2-4}{x+3}$

$$f'(x) = \frac{(x+3)(2x)-(x^2-4)(1)}{(x+3)^2} = \frac{2x^2+6x-x^2+4}{(x+3)^2} = \frac{x^2+6x+4}{(x+3)^2}$$

Set $x^2+6x+4=0$: Quadratic formula:

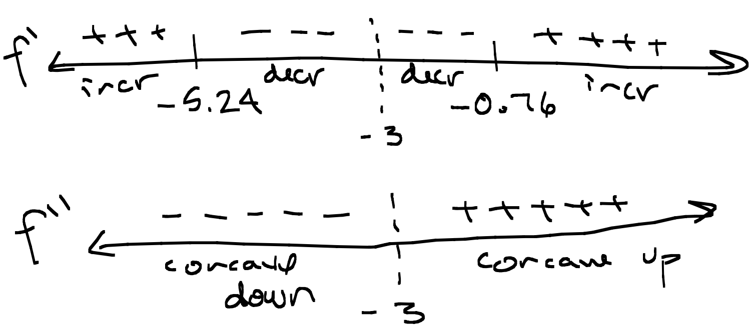
$$x = \frac{-6 \pm \sqrt{6^2-4(1)(4)}}{2(1)} = \frac{-6 \pm \sqrt{20}}{2}$$

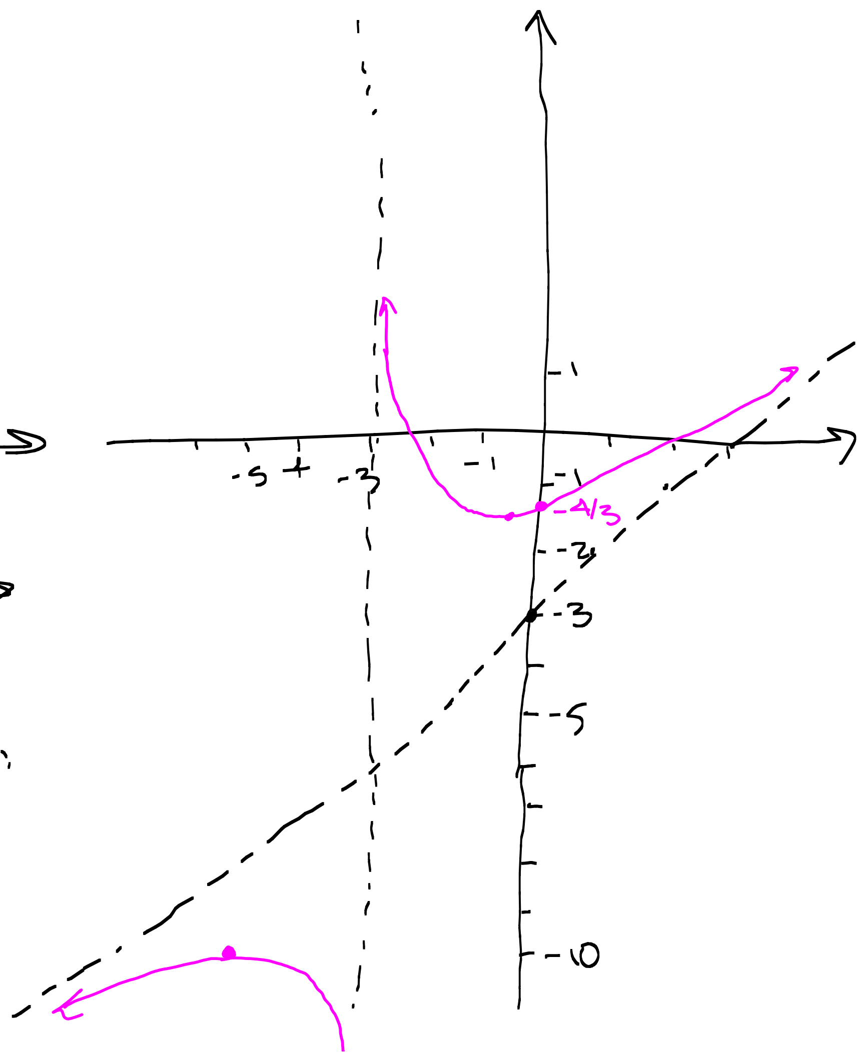
$$\approx -0.763,\ -5.236$$

$$f''(x) = \frac{(x+3)^2(2x+6)-(x^2+6x+4)(2)(x+3)(1)}{(x+3)^4}$$

$$= \frac{(x+3)\left[(x+3)(2x+6)-2(x^2+6x+4)\right]}{(x+3)^4}$$

$$= \frac{2\left[x^2+6x+9-x^2-6x-4\right]}{(x+3)^3} = \frac{10}{(x+3)^3}$$



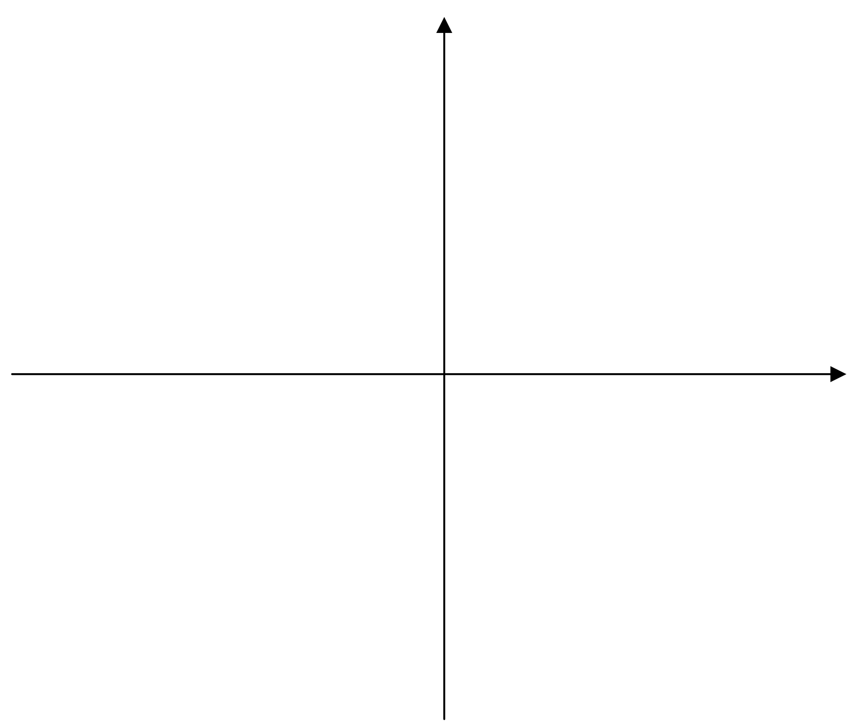


Find y-values for relative extrema:

$f(-5.24) \approx -10$

$f(-0.763) \approx -1.5$

Example 6: Sketch the graph of $f(x) = 5x^{2/3} - x^{5/3}$.



Example 7: Sketch the graph of $f(x) = x + \cos x$ on the interval $[-2\pi, 2\pi]$.