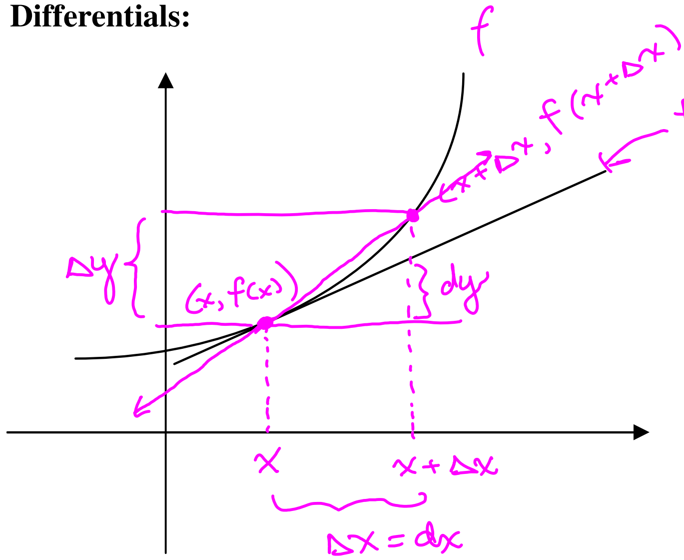
3.9: Differentials

Differentials:



$f'(x) = \frac{dy}{dx}$ = slope of tangent line

$\Delta y = f(x+\Delta x) - f(x)$

Let $dx = \Delta x$

$\frac{dy}{dx} = f'(x)$, so $dy = f'(x)dx$

If $y = f(x)$ is a differentiable function, we can let dx represent an amount of change in x.

Then the *differential* dy is defined to be $dy = f'(x)dx$. $dy \approx \Delta y$

dy is an approximation to $\Delta y = f(x+\Delta x) - f(x)$, which is the actual change in y.

Example 1: Find the differential dy for $y = \sqrt{6+x}$. Evaluate it when $x = 10$ and $dx = -0.3$. Compare it to the exact value of Δy.

$y = (6+x)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(6+x)^{-1/2}(1)$

$= \frac{1}{2\sqrt{6+x}}$

$\boxed{dy = \frac{1}{2\sqrt{6+x}}\,dx}$

$dy\Big|_{\substack{x=10\\ dx=-0.3}} = \frac{1}{2\sqrt{6+10}}(-0.3)$

$= \frac{1}{2(4)}(-0.3) = \frac{-0.3}{8} = \boxed{-0.0375}$

Compute actual change in y:

$x + dx = 10 - 0.3 = 9.7$

$\Delta y = f(x+\Delta x) - f(x) = f(9.7) - f(10)$

$= \sqrt{6+9.7} - \sqrt{6+10}$

$= \sqrt{15.7} - \sqrt{16}$

$= \sqrt{15.7} - 4$

$\approx \boxed{-0.037677}$

(compare to $dy = -0.0375$)

Note:

$\Delta x = dx = x_2 - x_1 = 9.7 - 10$

$= -0.3$

Example 2: Compute dy and Δy if $y = 5x + x^3$ as x changes from 3 to 3.05.

$\Delta x = dx = x_2 - x_1 = 3.05 - 3 = 0.05$

$\frac{dy}{dx} = 5 + 3x^2$

$dy = (5+3x^2)\,dx$

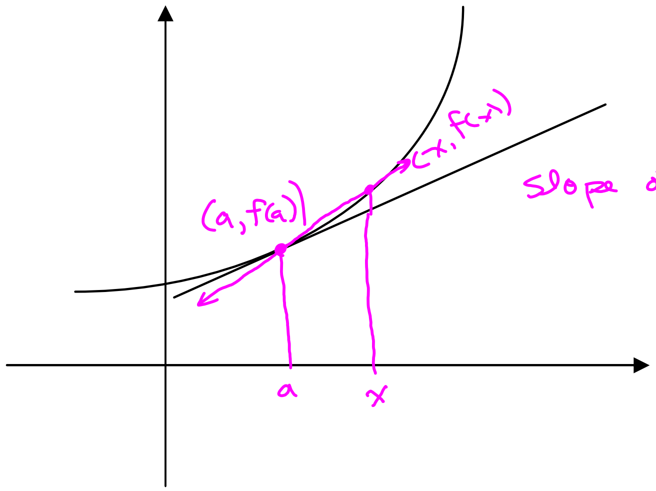
$dy\Big|_{\substack{x=3\\ dx=0.05}} = (5 + 3(3)^2)(0.05) = 32(0.05) = \boxed{1.6}$

$\Delta y = y_2 - y_1 = f(3.05) - f(3) = (5(3.05) + (3.05)^3) - (5(3) + 3^3)$

$= 43.622125 - 42 = \boxed{1.622625}$

(using a tangent line to approximate the function near a particular x-value)

The linearization of a function:



slope of secant line $= \dfrac{f(x)-f(a)}{x-a}$

$\approx f'(a) =$ slope of tangent line

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$f'(a) \approx \frac{f(x)-f(a)}{x-a} \Rightarrow f'(a)(x-a) \approx f(x)-f(a)$$

$$f(x)-f(a) \approx f'(a)(x-a)$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

This tangent line gives us an approximation for the value of the function near *a*.

> The *linearization* of *f* at *a* is
>
> $L(x) = f(a) + f'(a)(x-a)$
>
> The approximation $f(x) \approx L(x)$ is the standard linear approximation of *f* at *a*.

Example 3: Find the linearization of $f(x) = x^4$ at 3. Use this to approximate $(3.013)^4$ and $(2.999)^4$.

Find the equation of the tangent line:

$f'(x) = 4x^3$

$m = f'(3) = 4(3)^3 = 4(27) = 108$

Find y-value: $f(3) = 3^4 = 81$

$y - y_1 = m(x - x_1)$

$y - 81 = 108(x-3)$

$y - 81 = 108x - 324$

$y = 108x - 243$

Linearization is

$\boxed{L(x) = 108x - 243}$

Approximation of $(3.013)^4$:

$L(3.013) = 108(3.013) - 243$

$= \boxed{82.404}$

Approximation of $(2.999)^4$:

$L(2.999) = 108(2.999) - 243$

$= \boxed{80.892}$

Example 4: Find the linearization of $f(x) = \sin x$ at $60°$. Use this to approximate $\sin 62°$ and $\sin 58°$.

Error propagation:

If x is the measured value of a variable and $x + \Delta x$ is the exact value of the variable, then Δx is the measurement error. If we use the measured value of x to calculate the value of a function f, then the propagated error is $\Delta y = f(x + \Delta x) - f(x)$. The propagated error can be estimated by calculating $dy = f'(x)dx \approx f'(x)\Delta x$.

<u>Estimating propagated error</u>:

If x is the measured value of a variable and Δx is the measurement error, then:

Estimated propagated error: $dy = f'(x)dx \approx f'(x)\Delta x$

Estimated relative error: $\dfrac{dy}{y}$

Example 5: The measurement of the radius of a circle is 20 inches with a maximum error of 0.10 inch. Approximate the maximum propagated error and the relative error in computing the area and the circumference of the circle.

Area

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$dA = 2\pi r\,dr$

Propogated error $= dA\Big|_{\substack{r = 20'' \\ dr = \Delta r = 0.10''}}$

$= 2\pi(20'')(0.10'')$

$= 4\pi \text{ in}^2$

Relative error $= \frac{dA}{A} = \frac{4\pi \text{ in}^2}{\pi(20\text{in})^2} = \frac{4\pi \text{ in}^2}{400\pi \text{ in}^2} = \frac{1}{100} = 0.01 \Rightarrow \boxed{1\% \text{ error}}$

Circumference

$C = 2\pi r$

$\frac{dC}{dr} = 2\pi$

$dC = 2\pi\,dr$

Max. Propogated error

$= dC\Big|_{\substack{r=20'' \\ dr=0.10''}}$

$= 2\pi(0.10'')$

$= 0.2\pi \text{ in}$

$\approx \boxed{0.628 \text{ in}}$

Relative error $= \frac{dC}{C} = \frac{0.2\pi \text{ in}}{2\pi(20\text{ in})}$

$= \frac{0.2\pi}{40\pi}$

$= \frac{2}{400} = \frac{1}{200}$

$= 0.005$

$\Rightarrow \boxed{0.5\%}$

Example 6: The measurements of the height and inside radius of a right cylinder are 50 feet and 30 feet, respectively. The maximum possible error in each measurement is about 3 inches per each 10 feet of measured length. Approximate the maximum propagated error and the relative error in computing the volume of the cylinder.

For h, measurement error $\Delta h = dh = 50\cancel{\text{ft}}\left(\frac{3\text{ in}}{10\cancel{\text{ft}}}\right) = 5(3)\text{ in} = 15\cancel{\text{in}}\left(\frac{1\text{ft}}{12\cancel{\text{in}}}\right) = 1.25\text{ ft}$

For r, measurement error $\Delta r = dr = 30\text{ ft}\left(\frac{3\text{ in}}{10\text{ ft}}\right) = 9\text{ in}\left(\frac{1\text{ft}}{12\text{ in}}\right) = 0.75\text{ ft}$

$V = \pi r^2 h = (\pi r^2)(h)$

$\frac{dV}{dr} = (\pi r^2)\frac{d}{dr}(h) + (h)\frac{d}{dr}(\pi r^2) = \pi r^2\frac{dh}{dr} + h(2\pi r)$

multiply both sides by dr: $dV = \pi r^2 dh + 2\pi r h\,dr$

$dV\Big|_{\substack{h=50' \\ r=30' \\ dh=1.25' \\ dr=0.75'}} = \pi(30\text{ ft})^2(1.25\text{ ft}) + 2\pi(30\text{ft})(50\text{ft})(0.75\text{ ft})$

$= 1125\pi \text{ ft}^3 + 2250\pi \text{ ft}^3 = \boxed{3375\pi \text{ ft}^3}$

$\approx$ max. propagated error

Relative error $= \frac{dV}{V} = \frac{3375\pi \text{ ft}^3}{\pi(30')^2(50')} = \frac{3375\pi \text{ ft}^3}{45000\pi \text{ ft}^3}$

$= \frac{3375}{45000} = 0.075$

$\Rightarrow \boxed{7.5\% \text{ error}}$