4.4: The Fundamental Theorem of Calculus

Evaluating the area under a curve by calculating the areas of rectangles, adding them up, and letting taking the limit as $n \to \infty$ is okay in theory but is tedious at best and not very practical.

Fortunately, there is a theorem that makes calculating the area under the curve (definite integral) much easier.

> The Fundamental Theorem of Calculus:
>
> Let *f* be continuous on the interval $[a,b]$. Then,
>
> $$\int_a^b f(x)dx = F(b) - F(a)$$
>
> where *F* is any antiderivative of *f*; in other words, where $F'(x) = f(x)$.

Notation: We'll use this notation when evaluating definite integrals.

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example 1: Find the area under the graph of $f(x) = x$ between 0 and 3.

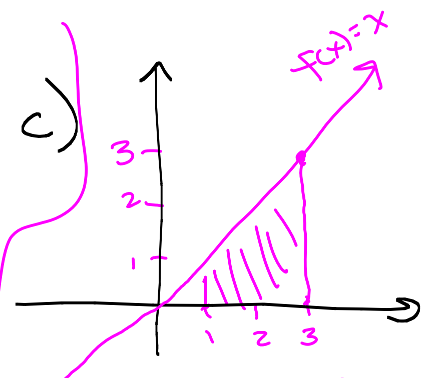
$f(x) = x$

Antiderivative: $F(x) = \int x\,dx = \frac{x^2}{2} + C$

$$\text{Area} = \int_0^3 x\,dx = \left(\frac{x^2}{2} + C\right)\Big|_0^3 = \frac{3^2}{2} + C - \left(\frac{0^2}{2} + C\right)$$

$$= \frac{9}{2} + C - 0 - C$$

$$= \boxed{\frac{9}{2}}$$



Area $= \frac{1}{2}(3)(3) = \frac{9}{2}$
(using geometry)

Alternate notation (used in Larson book)

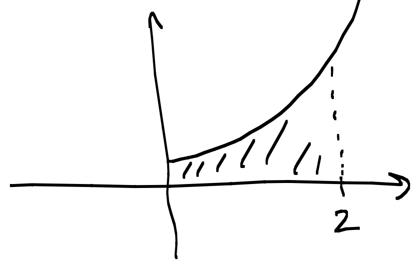
$$\int_0^3 x\,dx = \left[\frac{x^2}{2} + C\right]_0^3 = \frac{3^2}{2} + C - \left(\frac{0^2}{2} + C\right) = \boxed{\frac{9}{2}}$$

Notice that the constant C disappeared when we evaluated the definite integral. This will always happen.

$$\int_a^b f(x)dx = \left(F(x)+c\right)\Big|_a^b = \left(F(b)+c\right)-\left(F(a)+c\right) = F(b)+c-F(a)-c = F(b)-F(a)$$

So from now on, we'll omit the "$+c$" when evaluating definite integrals.

Example 2: Find the area under the graph of $f(x)=4x^2+1$ over the interval $[0,2]$. (Compare with our approximation in Section 4.2, Example 5).



$$\text{Area} = \int_0^2 (4x^2+1)\,dx = \left(\frac{4x^3}{3}+x\right)\Big|_0^2$$

$$= \frac{4(2)^3}{3}+2-\left(\frac{4(0)^3}{3}+0\right)$$

$$= \frac{32}{3}+2-0 = \frac{32}{3}+\frac{6}{3} = \boxed{\frac{38}{3}} = \boxed{12\tfrac{2}{3}}$$

From Ex5 in 4.2
Right Endpts: 17
Left Endpts: 9
Midpts: 12.5

← approximations of the area

Example 3: Evaluate $\int_{-2}^{4}(3x^2-x+4)dx$.

$$\int_{-2}^{4}(3x^2-x+4)\,dx = \left(\frac{3x^3}{3}-\frac{x^2}{2}+4x\right)\Big|_{-2}^{4} = \left(x^3-\frac{x^2}{2}+4x\right)\Big|_{-2}^{4}$$

$$= \left[(4)^3-\frac{4^2}{2}+4(4)\right]-\left[(-2)^3-\frac{(-2)^2}{2}+4(-2)\right] = \left[64-8+16\right]-\left[-8-2-8\right]$$

$$= \left[64+8\right]-\left[-18\right] = 72+18 = \boxed{90}$$

Check my antiderivative

$$\frac{d}{dx}\left(x^3-\frac{1}{2}x^2+4x\right) = 3x^2-\frac{1}{2}(2x)+4$$
$$= 3x^2-x+4 \quad \checkmark \text{ok}$$

Example 4: Evaluate $\int_0^{\pi}(4x^3+\cos x)dx$.

$$\int_0^{\pi}(4x^3+\cos x)dx = \left(\frac{4x^4}{4}+\sin x\right)\Big|_0^{\pi} = (x^4+\sin x)\Big|_0^{\pi}$$

$$= (\pi^4+\sin\pi)-(0^4+\sin 0) = \pi^4+0-0-0 = \boxed{\pi^4}$$

Example 5: Evaluate $\int_1^3\left(\frac{3}{t^2}\right)dt$.

$$\int_1^3 3t^{-2}dt = \frac{3t^{-1}}{-1}\Big|_1^3 = -\frac{3}{t}\Big|_1^3 = -\frac{3}{3}-\left(-\frac{3}{1}\right)$$

$$= -1+3 = \boxed{2}$$

Example 6: Evaluate $\int_2^9\frac{1}{\sqrt{u}}du$.

$$\int_2^9 u^{-1/2}du = \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Big|_2^9 = \frac{u^{1/2}}{1/2}\Big|_2^9 = 2u^{1/2}\Big|_2^9 = 2\sqrt{u}\Big|_2^9$$

$$= 2\sqrt{9}-2\sqrt{2} = 2(3)-2\sqrt{2} = \boxed{6-2\sqrt{2}}$$

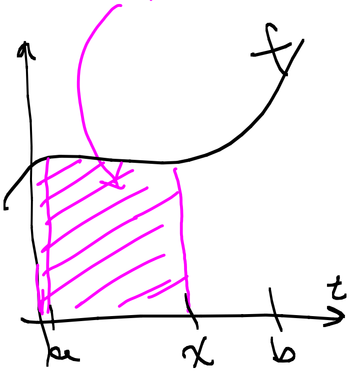
Example 7: Evaluate $\int_{-2}^4\frac{1}{x^3}dx$

$\int_{-2}^4\frac{1}{x^3}dx$ is an <u>improper integral</u>. There is a discontinuity in $[-2,4]$ (at $x=0$), so we can't apply the Fun. Theorem of Calculus.

Some improper integrals can be evaluated. We'll do it in Calculus 2.

For now, if f has an infinite discontinuity anywhere in $[a,b]$, assume that $\int_a^b f(x)\,dx$ does not exist. Some of these integrals do exist….you will learn how to handle such integrals in Calculus 2.

$\text{Area} = g(x) = \int_a^x f(t)\,dt$



<u>The Fundamental Theorem of Calculus, Part II</u>:

Let f be continuous on the interval $[a,b]$. Then the function g defined by

$$g(x) = \int_a^x f(t)\,dt\,, \qquad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

In other words, $\dfrac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$.

<u>Example 1:</u> Find the derivative of the function $f(x) = \int_3^x \dfrac{t^2 - 2t + 4}{t-2}\,dt$.

$$f(x) = \int_3^x \frac{t^2-2t+4}{t-2}\,dt$$

$$f'(x) = \boxed{\frac{x^2-2x+4}{x-2}}$$

<u>Example 2:</u> Find $\dfrac{d}{dx}\int_{-2}^{\sin x} \sqrt{t^4+2}\,dt$.

$\text{Area} = A = \int_{-2}^{\sin x} \sqrt{t^4+2}\,dt$. I want to find $\dfrac{dA}{dx}$.

Let $u = \sin x$

Then $A = \int_{-2}^{u} \sqrt{t^4+2}\,dt$

$$\frac{dA}{du} = \sqrt{u^4+2}$$

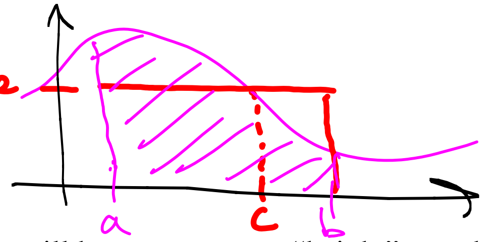
$u = \sin x$

$\dfrac{du}{dx} = \cos x$

Chain Rule: $\dfrac{dA}{dx} = \dfrac{dA}{du}\cdot\dfrac{du}{dx} = \sqrt{u^4+2}\,(\cos x) = \boxed{\sqrt{\sin^4 x + 2}\,(\cos x)}$

$\boxed{\cos x\sqrt{\sin^4 x + 2}}$

Area of rectangle $= f_{ave}(b-a) = \int_a^b f(x)\,dx$ $f(c) = f_{ave}$



The mean (average) value of a function:

On the interval $[a,b]$, a continuous function $f(x)$ will have an average "height" c such that the rectangle with width $b-a$ and height c will have the same area as the area under the curve over $[a,b]$. This c is the *average value of the function f over* $[a,b]$.

<u>Mean Value Theorem for Integrals</u>:

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

This number c is called the *average value* of the function f on the interval $[a,b]$.

The *average value* of a continuous function f on the interval $[a,b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

<u>Example 8:</u> Find the average value of the function $f(x) = 4x^3 - x^2$ over the interval $[-3,2]$.

Avg value $= \frac{1}{2-(-3)}\int_{-3}^{2}(4x^3 - x^2)\,dx = \frac{1}{5}\left[\frac{4x^4}{4} - \frac{x^3}{3}\right]\Big|_{-3}^{2}$

$= \frac{1}{5}\left[x^4 - \frac{x^3}{3}\right]\Big|_{-3}^{2} = \frac{1}{5}\left[(2)^4 - \frac{2^3}{3} - \left((-3)^4 - \frac{(-3)^3}{3}\right)\right]$

$= \frac{1}{5}\left[16 - \frac{8}{3} - \left(81 + \frac{27}{3}\right)\right] = \frac{1}{5}\left[16 - \frac{8}{3} - 81 - 9\right] = \frac{1}{5}\left[16 - \frac{8}{3} - 90\right]$

$= \frac{1}{5}\left[-\frac{8}{3} - 74\right] = \frac{1}{5}\left[-\frac{8}{3} - \frac{222}{3}\right]$

$= \frac{1}{5}\left[-\frac{230}{3}\right]$

$= -\frac{230}{15}$

$= \boxed{-\frac{46}{3}}$

<u>Example 9:</u> Determine the average value of $f(x) = \sin x$ on the interval $[0,\pi]$.

Avg value $= \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}(-\cos x)\Big|_0^{\pi}$

$= -\frac{1}{\pi}\cos x\Big|_0^{\pi} = -\frac{1}{\pi}\cos(\pi) - \left(-\frac{1}{\pi}\cos(0)\right)$

$= -\frac{1}{\pi}(-1) + \frac{1}{\pi}(1) = \frac{1}{\pi} + \frac{1}{\pi} = \boxed{\frac{2}{\pi}}$

Homework Questions

4.2 #17

$$\sum_{i=1}^{20}(i-1)^2 = \sum_{i=1}^{20}(i^2-2i+1)$$

$$= \sum_{i=1}^{20} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{20} 1$$

$$= \frac{20(20+1)(2(20)+1)}{6} - \cancel{2}\left(\frac{20(20+1)}{\cancel{2}}\right) + 1(20)$$

$$= \frac{\overset{10}{\cancel{20}}(\overset{7}{\cancel{21}})(41)}{\underset{\overset{3}{\cancel{6}}}{\cancel{6}}} - 20(21) + 20$$

$$= 10(7)(41) - 420 + 20$$

$$= 2470$$

Formulas we can use:

(p. 255, Theorem 4.2)

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

we'll use $n=20$

Calcchat has

$$\sum_{i=1}^{20}(i-1)^2 = \sum_{i=1}^{19} i^2$$

$$\underbrace{(1-1)^2}_{i=1} + \underbrace{(2-1)^2}_{i=2} + \underbrace{(3-1)^2}_{i=3} + \dots + \underbrace{(19-1)^2}_{i=19} + \underbrace{(20-1)^2}_{i=20}$$

$$0^2 + 1^2 + 2^2 + \dots + 18^2 + 19^2$$

$$= \sum_{k=1}^{19} k^2 = \frac{19(20)(2\cdot 19+1)}{6}$$

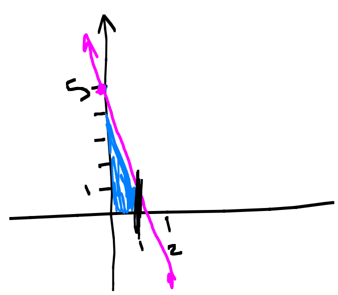
$$= \frac{19(20)(39)}{6} = 2470$$

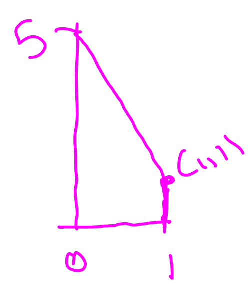
#45

$y = -4x+5$, $[0,1]$

$f(x)$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$





Right endpoints.

in the ith interval, the right endpoint is

$$c_i = a + i\Delta x$$

$$= 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$$

height: $f(c_i) = f\left(\frac{i}{n}\right) = -4\left(\frac{i}{n}\right)+5 = \frac{-4i}{n} + 5$

next page

$\#^{4.2}45$ cont'd) $\text{Area} = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(c_i)\Delta x\right)$

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{\infty}\left(-\frac{4i}{n}+5\right)\left(\frac{1}{n}\right)\right)$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(-\frac{4i}{n^2}+\frac{5}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[-\frac{4}{n^2}\sum_{i=1}^{n} i + \frac{5}{n}\sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty}\left[-\frac{4}{n^2}\cdot\frac{n(n+1)}{2} + \frac{5}{n}\cdot n\right]$$