4.5: Integration by Substitution

Most functions cannot be integrated using only the formulas we have learned so far. In Calculus II, you will learn several advanced integration techniques. For now, we'll learn just one new integration technique, called substitution.

Example 1: Find $\int 4(4x-9)^7\,dx$.

One way to do this would be to multiply it out into a long polynomial….YUK!

Here's another way:

$$\int \underbrace{4\,\underbrace{(4x-9)^7}_{u^7}\,dx}_{du} = \int u^7\,du = \frac{u^8}{8} + c = \boxed{\frac{(4x-9)^8}{8} + c}$$

Let $u = 4x-9$

$\frac{du}{dx} = 4$

$du = 4\,dx$

Check: $\frac{d}{dx}\left(\frac{1}{8}(4x-9)^8\right) = \frac{1}{\cancel{8}}(\cancel{8})(4x-9)^7(4)$

$= 4(4x-9)^7$ ✓ok

> The Substitution Rule:
>
> If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then
>
> $$\int f(g(x))g'(x)dx = \int f(u)du$$

Example 2: Find $\int 6x^2(2x^3+5)^4\,dx$.

$$\int \underbrace{6x^2\,\underbrace{(2x^3+5)^4}_{u^4}\,dx}_{du} = \int u^4\,du = \frac{u^5}{5} + c = \boxed{\frac{(2x^3+5)^5}{5} + c}$$

Check it!

$u = 2x^3+5$

$\frac{du}{dx} = 6x^2$

$du = 6x^2\,dx$

Example 3: Find $\int -2\sqrt{-2x-4}\;dx$.

$$\int -2\sqrt{-2x-4}\;dx = \int \underbrace{-2\,\underbrace{(-2x-4)^{1/2}}_{u^{1/2}}\,dx}_{du} = \int u^{1/2}\,du$$

$u = -2x-4$

$\frac{du}{dx} = -2$

$du = -2\,dx$

$$= \frac{u^{3/2}}{3/2} + c = \frac{2}{3}u^{3/2} + c = \boxed{\frac{2}{3}(-2x-4)^{3/2} + c}$$

Example 4: Find $\int x^3(x^4-1)^6\,dx$.

$$\int \underbrace{x^3}\underbrace{(x^4-1)^6}_{u^6}\underbrace{dx}_{\frac{1}{4}du} = \int \frac{1}{4}u^6\,du = \frac{1}{4}\int u^6\,du$$

$$= \frac{1}{4}\cdot\frac{u^7}{7} + c = \frac{1}{28}u^7 + c$$

$$= \boxed{\frac{1}{28}(x^4-1)^7 + c}$$

$u = x^4 - 1$

$\frac{du}{dx} = 4x^3$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

Check: $\frac{d}{dx}\left[\frac{1}{28}(x^4-1)^7\right] = \frac{1}{28}(7)(x^4-1)^6(4x^3)$

$= \frac{\cancel{28}}{\cancel{28}}(x^4-1)^6(x^3) = x^3(x^4-1)^6$ ✓

Example 5: Find $\int \sin 5t\,dt$.

$$\int \underbrace{\sin(5t)}_{\sin(u)}\underbrace{dt}_{\frac{1}{5}du} = \int \frac{1}{5}\sin(u)\,du = \frac{1}{5}\int \sin(u)\,du$$

$$= \frac{1}{5}(-\cos(u)) + c$$

$$= \boxed{-\frac{1}{5}\cos(5t) + c}$$

$u = 5t$

$\frac{du}{dt} = 5$

$du = 5\,dt$

$\frac{1}{5}du = dt$

Check: $\frac{d}{dt}\left(-\frac{1}{5}\cos(5t)\right) = -\frac{1}{\cancel{5}}(-\sin(5t))(\cancel{5})$

$= \sin(5t)$ ✓ ok

Example 6: Find $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$.

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int \underbrace{(\cos\sqrt{x})}_{\cos u}\underbrace{\left(\frac{1}{\sqrt{x}}\right)dx}_{2du}$$

$$= \int 2\cos u\,du = 2\sin u + c$$

$$= \boxed{2\sin\sqrt{x} + c}$$

$u = \sqrt{x} = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

Check: $\frac{d}{dx}\left(2\sin(x^{1/2})\right) = \cancel{2}(\cos(x^{1/2}))\left(\frac{1}{\cancel{2}}\right)x^{-1/2}$

$= \frac{\cos\sqrt{x}}{\sqrt{x}}$ ✓

Example 7: Find $\int \frac{x}{\sqrt[3]{5x^2-8}}\,dx$.

$u = 5x^2 - 8$

$\frac{du}{dx} = 10x$

$du = 10x\,dx$

$\frac{1}{10}\,du = x\,dx$

$$\int \frac{x}{\sqrt[3]{5x^2-8}}\,dx = \int x\left(\frac{1}{\sqrt[3]{5x^2-8}}\right)dx$$

$$= \int \underbrace{x\,\underbrace{(5x^2-8)^{-1/3}}_{u^{-1/3}}\,dx}_{\frac{1}{10}du} = \int \frac{1}{10}u^{-1/3}\,du = \frac{1}{10}\int u^{-1/3}du$$

$$= \frac{1}{10}\cdot\frac{u^{2/3}}{2/3} + c = \frac{1}{10}\cdot\frac{3}{2}u^{2/3} + c$$

$$= \boxed{\frac{3}{20}(5x^2-8)^{2/3} + c}$$

$$= \frac{3}{20}\sqrt[3]{(5x^2-8)^2} + c$$

Example 8: Find $\int \cos x \sin^5 x\,dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$$\int \cos x \sin^5 x\,dx = \int \underbrace{\cos x\,\underbrace{(\sin x)^5}_{u^5}\,dx}_{du}$$

$$= \int u^5 du = \frac{u^6}{6} + c$$

$$= \boxed{\frac{1}{6}(\sin(x))^6 + c}$$

Example 9: Find $\int \sec^2\left(\frac{x}{7}\right)dx$.

$u = \frac{x}{7} = \frac{1}{7}x$

$\frac{du}{dx} = \frac{1}{7}$

$du = \frac{1}{7}dx$

$7\,du = dx$

$$\int \underbrace{\sec^2\left(\tfrac{1}{7}x\right)}_{\sec^2(u)}\underbrace{dx}_{7du} = 7\int \sec^2(u)\,du$$

$$= 7\tan(u) + c$$

$$= \boxed{7\tan\left(\tfrac{1}{7}x\right) + c}$$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\Downarrow$

$\int \sec^2 u\,du = \tan u + c$

Check: $\frac{d}{dx}\left(7\tan\left(\tfrac{1}{7}x\right)\right)$

$= \cancel{7}\sec^2\left(\tfrac{1}{7}x\right)\left(\tfrac{1}{\cancel{7}}\right)$

$= \sec^2\left(\tfrac{1}{7}x\right)$ ✓

Example 10: Find $\int x(5+x)^4\,dx$.

$u = 5 + x$

$\frac{du}{dx} = 1$

$du = dx$

$u - 5 = x$

$$\int x\underbrace{(5+x)^4}_{u^4}\underbrace{dx}_{du} = \int \underbrace{x}_{u-5} u^4 du$$

$$= \int (u-5)(u^4)\,du$$

$$= \int (u^5 - 5u^4)\,du = \frac{u^6}{6} - \frac{\cancel{5}u^5}{\cancel{5}} + c = \frac{u^6}{6} - u^5 + c$$

$$= \boxed{\frac{1}{6}(5+x)^6 - (5+x)^5 + c}$$

Check: $\frac{d}{dx}\left(\frac{1}{6}(5+x)^6 - (5+x)^5\right) + c = \frac{1}{6}(6)(5+x)^5(1) - 5(5+x)^4(1)$

$= (5+x)^5 - 5(5+x)^4$

$= (5+x)^4[5+x-5]$

Definite integrals:

<u>The Substitution Rule for Definite Integrals</u>:

If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

There are two methods of evaluating definite integrals when substitution is involved:

<u>Method 1</u>: Find the antiderivative using substitution and switch back to your original variable. Then evaluate the definite integral using the original upper and lower limits.

<u>Method 2</u>: (Using the Substitution Rule for Definite Integrals) Find the antiderivative using substitution, but don't switch back to your original variable. Instead, calculate the upper and lower limits in terms of *u* (or whatever variable you used to substitute). Then evaluate your definite integral using these "new" upper and lower limits.

$u = 2x^5 - 8$

$\frac{du}{dx} = 10x^4$

$du = 10x^4 dx$

$\frac{1}{10} du = x^4 dx$

<u>Example 11:</u> Evaluate $\int_{-1}^{2} x^4(2x^5-8)^3 dx$ using both methods.

<u>Method 1</u>: $\int_{-1}^{2} \underbrace{x^4 (\underbrace{2x^5-8)^3}_{u^3} dx}_{\frac{1}{10}du} = \frac{1}{10}\int_{-1=x}^{2=x} u^3 du$

$= \frac{1}{10}\cdot\frac{u^4}{4}\Big|_{x=-1}^{x=2} = \frac{1}{40}u^4\Big|_{x=-1}^{x=2} = \frac{1}{40}(2x^5-8)^4\Big|_{-1=x}^{2=x}$

$= \frac{1}{40}(2(2^5)-8)^4 - \frac{1}{40}(2(-1)^5-8)^4 = \frac{56^4}{40} - \frac{(-10)^4}{40} = \frac{9834496}{40} - \frac{10000}{40}$

$= \frac{9824496}{40} = \boxed{245612.4}$

<u>Method 2</u>: (changing the limits of integration)

(Using the above theorem – Substitution Rule)

$\int_{-1}^{2} x^4(2x^5-8)^3 dx = \frac{1}{10}\int_{-10=u}^{56=u} u^3 du$

$= \frac{1}{10}\cdot\frac{u^4}{4}\Big|_{-10=u}^{56=u} = \frac{u^4}{40}\Big|_{u=-10}^{u=56} = \frac{(56)^4}{40} - \frac{(-10)^4}{40}$

$= \boxed{245612.4}$

$u = 2x^5 - 8$

$du = 10x^4 dx$

$\frac{1}{10} du = x^4 dx$

$x = -1 \Rightarrow u = 2(-1)^5 - 8$
$= 2(-1) - 8 = -10$

$x = 2 \Rightarrow u = 2(2)^5 - 8 = 64 - 8$
$= 56$

Ex $8\frac{1}{2}$

$$\int \sin(x)\underbrace{\cos(x)}_{u}\,dx = \int -u\,du$$

$\sin(x)\,dx = -du$

$$= -\int u\,du$$

$$= -\frac{u^2}{2} + C$$

$$= \boxed{-\frac{\cos^2 x}{2} + C}$$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

Check: $\frac{d}{dx}\left(-\frac{1}{2}\cos^2 x\right) = -\frac{1}{\cancel{2}}(\cancel{2})(\cos x)(-\sin x)$

$= \cos x \sin x$ ✓ OK

Another approach: $\int \underbrace{\sin x}_{u}\,\underbrace{\cos x\,dx}_{du}$

$u = \sin x$

$du = \cos x\,dx$

$$= \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{\sin^2 x}{2} + C}$$ (It checks!)

Are these answers equivalent?

$$-\frac{1}{2}\cos^2 x + C = -\frac{1}{2}(1 - \sin^2 x) + C = -\frac{1}{2} + \frac{1}{2}\sin^2 x + C$$

$$= \frac{1}{2}\sin^2 x - \frac{1}{2} + C$$

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$\sin^2\theta = 1 - \cos^2\theta$

$$= \frac{\sin^2 x}{2} \underbrace{- \frac{1}{2} + C}_{C_2}$$

Note: discontinuous where $3-5x=0$

$3=5x$

$\frac{3}{5}=x$ not in interval $[-4,-2]$

Example 12: Evaluate $\int_{-4}^{-2}\frac{4}{(3-5x)^3}\,dx$

$$4\int_{-4}^{-2}(3-5x)^{-3}\,dx = 4\left(-\frac{1}{5}\right)\int_{23}^{13}u^{-3}\,du$$

$$= -\frac{4}{5}\cdot\frac{u^{-2}}{-2}\Bigg|_{23}^{13} = \frac{4}{10u^2}\Bigg|_{23}^{13}$$

$$= \frac{2}{5u^2}\Bigg|_{23}^{13} = \frac{2}{5(13)^2}-\frac{2}{5(23)^2}$$

$$= \frac{144}{89401} \approx \boxed{0.0016107}$$

$u=3-5x$

$\frac{du}{dx}=-5$

$du=-5\,dx$

$-\frac{1}{5}\,du=dx$

$x=-4 \Rightarrow u=3-5(-4)$
$=23$

$x=-2 \Rightarrow u=3-5(-2)$
$=13$

Example 13: Evaluate $\int_{\pi/4}^{\pi/3}\frac{\sin x}{\cos^3 x}\,dx$.

$$\int_{\frac{\pi}{4}}^{\pi/3}\underbrace{\sin x}_{-du}\,\underbrace{(\cos x)^{-3}}_{u^{-3}}\,dx = -\int_{\sqrt{2}/2}^{1/2}u^{-3}\,du$$

$$= -\frac{u^{-2}}{-2}\Bigg|_{\sqrt{2}/2}^{1/2} = \frac{1}{2u^2}\Bigg|_{\sqrt{2}/2}^{1/2}$$

$$= \frac{1}{2\left(\frac{1}{2}\right)^2}-\frac{1}{2\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{1}{2\left(\frac{1}{4}\right)}-\frac{1}{2\left(\frac{2}{4}\right)}$$

$$= \frac{1}{1/2}-\frac{1}{1} = 2-1 = \boxed{1}$$

$u=\cos x$

$\frac{du}{dx}=-\sin x$

$du=-\sin x\,dx$

$-du=\sin x$

$x=\frac{\pi}{4} \Rightarrow u=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

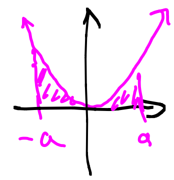
$x=\frac{\pi}{3} \Rightarrow u=\cos\frac{\pi}{3}=\frac{1}{2}$

Symmetry and definite integrals:

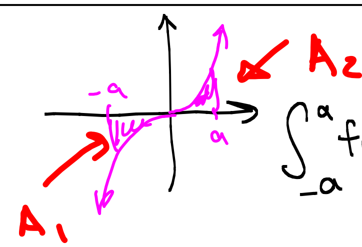
Suppose f is continuous on $[-a,a]$.

(a) If f is even (symmetric about ~~x-axis~~ y-axis, $f(-x)=f(x)$), then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$



(b) If f is odd (symmetric about origin, $f(-x)=-f(x)$), then $\int_{-a}^{a} f(x)dx = 0$.



$$\int_{-a}^{a} f(x)\,dx = -A_1 + A_2 = 0$$

Example 14: $\int_{-3}^{3} x^2\,dx$

$$\int_{-3}^{3} x^2 dx = 2\int_{0}^{3} x^2 dx$$

$$= 2\left(\frac{x^3}{3}\right)\Big|_0^3 = 2\left(\frac{3^3}{3}\right) - 2\left(\frac{0^3}{3}\right) = \boxed{18}$$

Example 15: Find $\int_{-13}^{13} 7x^5 dx$ $= 0$

Homework Qs

4.4 # 91) $F(x) = \int_0^{x^3} \sin(t^2)\,dt$. Find $F'(x)$

From Theorem 4.11 (Fundamental Theorem of Calculus)

If $G(x) = \int_0^x \sin(t^2)\,dt$, then $G'(x) = \sin(x^2)$.

$G(x) = \int_a^x f(t)\,dt$, then $G'(x) = f(x)$.

Back to # 91:

$$A = F(x) = \int_0^{x^3} \sin(t^2)\,dt$$

I want to find $\frac{dA}{dx}$ (which is $F'(x)$)

Let $u = x^3$.

$\frac{du}{dx} = 3x^2$

$$A = F(x) = \int_0^u \sin(t^2)\,dt$$

$$\frac{dA}{du} = \sin(u^2)$$

$$\frac{dA}{dx} = \frac{dA}{du} \cdot \frac{du}{dx} = \underbrace{\left(\sin(u^2)\right)}_{\frac{dA}{du}} \underbrace{\left(3x^2\right)}_{\frac{du}{dx}}$$

$$= \left(\sin\left((x^3)^2\right)\right)\left(3x^2\right)$$

$$= \boxed{3x^2 \sin(x^6)}$$

4.4 # 97(b) $s'(t) = v(t) = t^3 - 10t^2 + 27t - 18, \quad 1 \le t \le 7$

Position

$$s(t) = \int s'(t)\,dt = \int v(t)\,dt$$
$$= \int (t^3 - 10t^2 + 27t - 18)\,dt$$
$$= \frac{t^4}{4} - \frac{10t^3}{3} + \frac{27t^2}{2} - 18t + C$$

Displacement: How far is the object from its starting point? so displacement $= s(b) - s(a)$

$$\text{Displacement} = \int_1^7 (t^3 - 10t^2 + 27t - 18)\,dt$$
$$= \left(\frac{t^4}{4} - \frac{10t^3}{3} + \frac{27t^2}{2} - 18t\right)\Bigg|_1^7$$
$$= \left(\frac{7^4}{4} - \frac{10(7)^3}{3} + \frac{27(7)^2}{2} - 18(7)\right) - \left(\frac{1^4}{4} - \frac{10(1)^3}{3} + \frac{27(1)^2}{2} - 18(1)\right)$$
$$= \boxed{0}$$

(b)

Total Distance:

$$\text{Distance} = \int_1^7 |v(t)|\,dt$$

I want to sketch $v(t) = t^3 - 10t^2 + 27t - 18$

Possible rational roots: $\pm 1, 3, 2, 9, 6, 18$

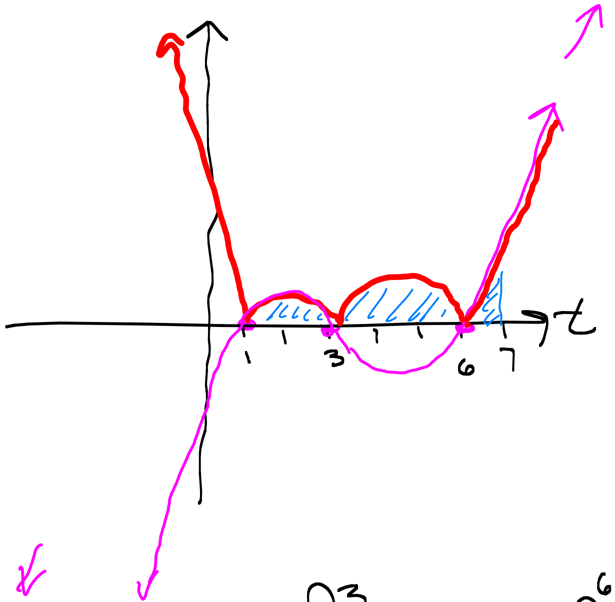
$$\begin{array}{r|rrrr} 1 & 1 & -10 & 27 & -18 \\ & & 1 & -9 & 18 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$ Remainder

$$\begin{array}{r} t^2 - 9t + 18 \\ t-1 \overline{)\, t^3 - 10t^2 + 27t - 18} \\ -(t^3 - t^2) \\ \hline -9t^2 + 27t \\ -(-9t^2 + 9t) \\ \hline 18t - 18 \\ 18t - 18 \\ \hline 0 \end{array}$$

So $v(t) = (t-1)(t^2 - 9t + 18)$
$= (t-1)(t-3)(t-6)$

t-intercepts: 1, 3, 6

See next page



$$v(t) = t^3 - 10t^2 + 27t - 18$$

$$v(t) = (t-1)(t^2 - 9t + 18)$$

$$= (t-1)(t-3)(t-6)$$

Degree: odd

sign of leading term: (+)

Ends

$$\text{Distance} = \int_1^3 v(t)\,dt + \int_3^6 (-v(t))\,dt + \int_6^7 v(t)\,dt$$

$$= \int_1^3 v(t)\,dt - \int_3^6 v(t)\,dt + \int_6^7 v(t)\,dt$$

$$= s(3) - s(1) - \big(s(6) - s(3)\big) + s(7) - s(6)$$

$$= s(3) - s(1) - s(6) + s(3) + s(7) - s(6)$$

$$= 2s(3) - s(1) - 2s(6) + s(7)$$