5.1: The Natural Logarithmic Function: Differentiation

An algebraic approach to logarithms:

<u>Definition</u>: $\log_b x = y$ is equivalent to $b^y = x$.

The functions $f(x) = b^x$ and $g(x) = \log_b x$ are inverses of each other.
b is called the *base* of the logarithm.

The logarithm of base e is called the *natural logarithm*, which is abbreviated "ln".

<u>The natural logarithm</u>:

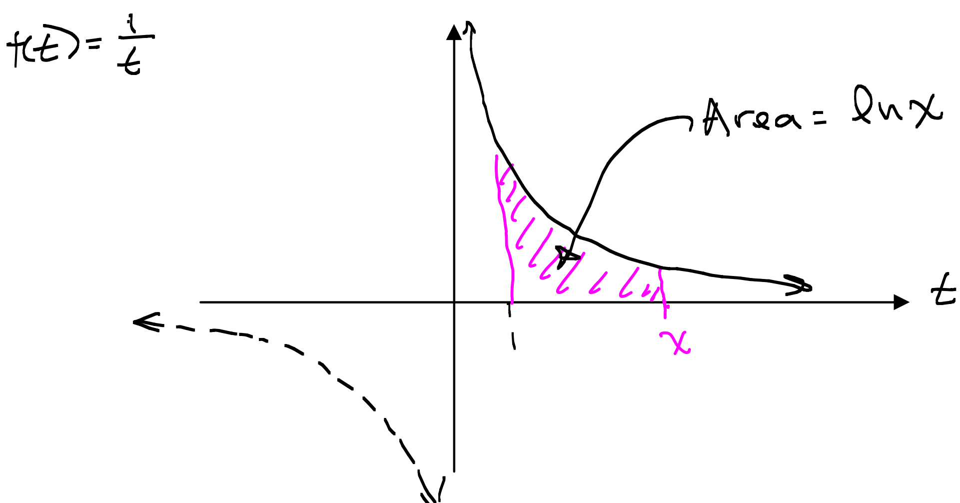
$$\ln x = \log_e x .$$

Therefore $\ln x = y$ is equivalent to $e^y = x$ and the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverses of each other.

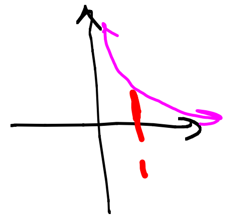
A calculus approach to the natural logarithm:

The natural logarithm function is defined as

$$\ln x = \int_1^x \frac{1}{t}\,dt , \quad x > 0 .$$



For $x > 1$, $\ln x$ can be interpreted as the area under the graph of $y = \dfrac{1}{t}$ from $t = 1$ to $t = x$.



Note: The integral is not defined for $x < 0$.

For $x = 1$, $\ln x = \int_1^1 \frac{1}{t}\,dt = 0$. So $\ln(1) = 0$

For $x < 1$, $\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$.

Recall:

> The Fundamental Theorem of Calculus, Part II:
>
> Let f be continuous on the interval $[a,b]$. Then the function g defined by
>
> $$g(x) = \int_a^x f(t)\,dt\,, \qquad a \le x \le b$$
>
> is continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) = f(x)$.

Apply the Fundamental Theorem of Calculus to the function $f(t) = \frac{1}{t}$.

$$\frac{d}{dx}\left(\underbrace{\int_1^x \frac{1}{t}\,dt}_{\ln x}\right) = \frac{1}{x}$$

This means that $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

> The Derivative of the Natural Logarithmic Function
>
> $$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Remember this!

Laws of Logarithms:

If x and y are positive numbers and r is a rational number, then:

1. $\ln(xy) = \ln x + \ln y$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

   Note: This also gives us $\ln\left(\frac{1}{x}\right) = -\ln x$.

   $$\ln\left(\frac{1}{x}\right) = \ln(1) - \ln(x) = 0 - \ln(x) = -\ln(x)$$

3. $\ln\left(x^r\right) = r \ln x$

Example 1: Expand $\ln\left(\frac{x^3\sqrt{x+5}}{x^2+4}\right)$.

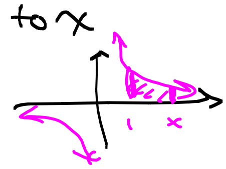
$$\ln\left(\frac{x^3(x+5)^{1/2}}{(x^2+4)(x-3)}\right) = \ln\left[x^3(x+5)^{1/2}\right] - \ln\left[(x^2+4)(x-3)\right]$$

$$= \ln(x^3) + \ln(x+5)^{1/2} - \left[\ln(x^2+4) + \ln(x-3)\right]$$

$$= \ln(x^3) + \ln(x+5)^{1/2} - \ln(x^2+4) - \ln(x-3)$$

$$= \boxed{3\ln(x) + \tfrac{1}{2}\ln(x+5) - \ln(x^2+4) - \ln(x-3)}$$

The graph of $y = \ln x$**:** $y = \ln x$ is area under graph of $f(t) = \frac{1}{t}$ from 1 to x



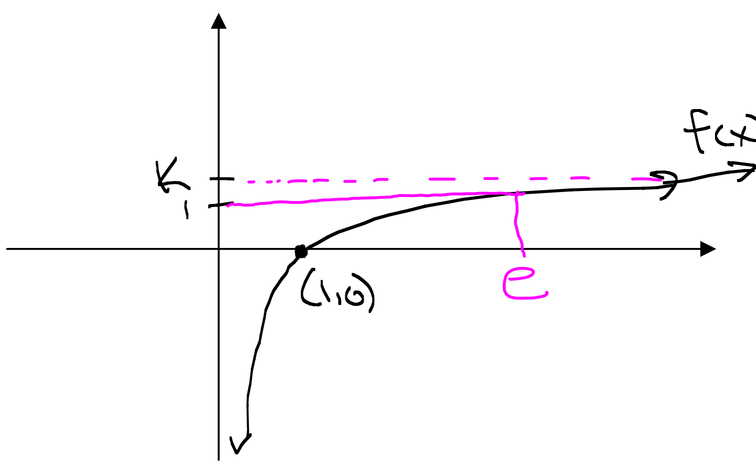
It can be shown that $\lim_{x\to\infty} \ln x = \infty$ and that $\lim_{x\to 0^+} \ln x = -\infty$.

For $x > 0$, $\frac{dy}{dx} = \frac{1}{x} > 0$ so $y = \ln x$ is increasing on $(0, \infty)$.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = -1x^{-2}$$

For $x > 0$, $\frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$ so $y = \ln x$ is concave down on $(0, \infty)$.

Recall: $\ln(1) = 0$ so the point $(1,0)$ is on the graph of $f(x) = \ln x$



Because $\ln 1 = 0$ and $y = \ln x$ is increasing to arbitrarily large values $\left(\lim_{x\to\infty} \ln x = \infty\right)$, the Intermediate Value Theorem guarantees that there is a number x such that $\ln x = 1$. That number is called *e*.

$e \approx 2.71828182845904523536$

(*e* is in irrational number—it cannot be written as a decimal that ends or repeats.)

So, $\ln(e) = 1$

Recall $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Ex $2\frac{1}{2}$: $y = (2x^5+3x)^4$

$$\frac{dy}{dx} = 4(2x^5+3x)^3(10x^4+3)$$

Ex $2\frac{3}{4}$: $y = \cos(2x^5+3x)$

$$\frac{dy}{dx} = [-\sin(2x^5+3x)](10x^4+3)$$

Example 2: Find $\frac{dy}{dx}$ for $y = \ln(2x^5+3x)$.

$$\frac{dy}{dx} = \frac{1}{2x^5+3x}\frac{d}{dx}(2x^5+3x)$$

$$= \frac{1}{2x^5+3x}(10x^4+3) = \boxed{\frac{10x^4+3}{2x^5+3x}}$$

Note: $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ or, written another way, $\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}$.

Example 3: Determine $\frac{d}{dx}(\ln(\cos x))$.

$$\frac{d}{dx}(\ln(\cos x)) = \frac{1}{\cos x}\frac{d}{dx}(\cos x)$$

$$= \frac{1}{\cos x}(-\sin x) = -\frac{\sin x}{\cos x} = \boxed{-\tan x}$$

Example 4: Find the derivative of $f(x) = \frac{1}{\ln x}$.

$$f(x) = (\ln x)^{-1}$$

$$f'(x) = -1(\ln x)^{-2}\frac{d}{dx}(\ln x) = -1(\ln x)^{-2}\left(\frac{1}{x}\right) = \boxed{-\frac{1}{x(\ln x)^2}}$$

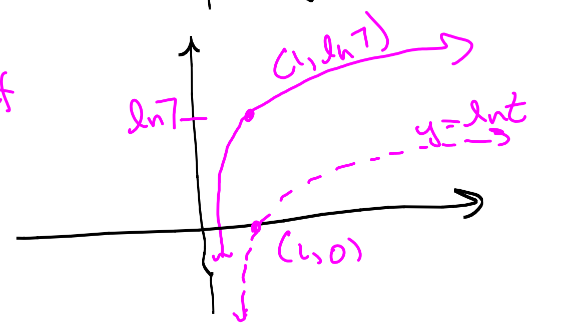
Example 5: Find the derivative of $f(x) = x^2 \ln x$.

Example 6: Find the derivative of $y = \frac{\ln x}{4x}$.

Example 7: Find the derivative of $g(t) = \ln(7t)$.

$$g'(t) = \frac{1}{7t}\frac{d}{dt}(7t) = \frac{1}{\cancel{7}t}(\cancel{7}) = \boxed{\frac{1}{t}}$$

Graph $y = \ln(t) + \ln(7)$



Note: $g(t) = \ln(7t) = \ln(7) + \ln(t)$
$= \ln(t) + \ln(7)$ constant

Example 8: Determine the derivative of $f(x) = \dfrac{\ln 6x}{(x+4)^5}$.

Logarithmic differentiation:

To differentiate $y = f(x)$:

1. Take the natural logarithm of both sides.

2. Use the laws of logarithms to expand.

3. Differentiate implicitly with respect to *x*.

4. Solve for $\dfrac{dy}{dx}$.

Example 9: Use logarithmic differentiation to find the derivative of

$$y = (x^2+2)^5(2x+1)^3(6x-1)^2.$$

Take ln of both sides: $\ln(y) = \ln\left[(x^2+2)^5(2x+1)^3(6x-1)^2\right]$

Apply log properties: $\ln y = 5\ln(x^2+2) + 3\ln(2x+1) + 2\ln(6x-1)$

Implicit diff: $\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[5\ln(x^2+2) + 3\ln(2x+1) + 2\ln(6x-1)\right]$

$$\frac{1}{y}\frac{dy}{dx} = 5\left(\frac{1}{x^2+2}\right)(2x) + 3\left(\frac{1}{2x+1}\right)(2) + 2\left(\frac{1}{6x-1}\right)(6)$$

Clean up and solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = y\left[\frac{10x}{x^2+2} + \frac{6}{2x+1} + \frac{12}{6x-1}\right]$

Put original function in for y: $\boxed{\frac{dy}{dx} = (x^2+2)^5(2x+1)^3(6x-1)^2\left[\frac{10x}{x^2+2} + \frac{6}{2x+1} + \frac{12}{6x-1}\right]}$

Example 10: Find y' for $y = \frac{(x^3+1)^4\sin^2 x}{\sqrt[3]{x}}$.

(Note: not worked during class)

$$y = \frac{(x^3+1)^4(\sin x)^2}{x^{1/3}}$$

$$\ln y = \ln\left[\frac{(x^3+1)^4(\sin x)^2}{x^{1/3}}\right]$$

$$\ln y = \ln(x^3+1)^4 + \ln(\sin x)^2 - \ln(x^{1/3})$$

$$\ln y = 4\ln(x^3+1) + 2\ln(\sin x) - \frac{1}{3}\ln(x)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(4\ln(x^3+1) + 2\ln(\sin x) - \frac{1}{3}\ln(x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = 4\left(\frac{1}{x^3+1}\right)(3x^2) + 2\left(\frac{1}{\sin x}\right)(\cos x) - \frac{1}{3}\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = y\left[\frac{12x^2}{x^3+1} + 2\cot x - \frac{1}{3x}\right]$$

$$\boxed{y' = \frac{dy}{dx} = \frac{(x^3+1)^4\sin^2 x}{\sqrt[3]{x}}\left(\frac{12x^2}{x^3+1} + 2\cot x - \frac{1}{3x}\right)}$$